Name: **Enrolment No:**	UPES UNIVERSITY WITH A PURPOSE

UNIVERSITY OF PETROLEUM AND ENERGY STUDIES
End Semester Examination, DECEMBER 2021

Course: Theory of real functions **Semester: III**
Program: B.Sc. (Hons.) Mathematics **Time: 03 hrs.**
Course Code: MATH 2010 **Max. Marks: 100**
Instructions: All questions are compulsory.

SECTION A (Each question carries 4 marks)

S. No.		**Marks**
Q1	Consider the function $f(x) = \begin{cases} 1 & if\ x = \frac{1}{n},\ where\ n \in \mathbb{N} \\ 0 & otherwise \end{cases}$ Then find $\lim_{x\to 0} f(x)$.	**CO1**
Q2	Give one example in support of each of the following statements- a. Let A be a nonempty subset of $\mathbb{R}$,such that the derived set A' of A is empty. Then there exists a function $f: A \to \mathbb{R}$ which is continuous. b. Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous function and A is a bounded subset of $\mathbb{R}$ then $f(A)$ is bounded.	**CO2**
Q3	Find the Taylor's polynomial of degree 6 for the function $cos\ x$ about $x = \frac{\pi}{4}$.	**CO4**
Q4	Find the value of c of Cauchy's mean value theorem for the functions $f(x) = x^3, g(x) = x^2$ in the interval [1,2].	**CO4**
Q5	Consider the following function defined on the interval $[a, b]$ $$f(x) = \begin{cases} \frac{1}{q}, & if\ x = \frac{p}{q}, p \in \mathbb{Z}, q \in \mathbb{N}, q > 0\ and\ gcd(p,q) = 1 \\ 0 & otherwise \end{cases}$$ Find the points of local minima of $f(x)$.	**CO3**

SECTION B (Each question carries 10 marks)

Q6	Prove that Thomae's function is continuous at $\mathbb{R}\backslash\mathbb{Q}$ but discontinuous at $\mathbb{Q}$.	**CO2**
Q7	Consider the set $S = [0,1]\backslash\left(\frac{1}{3}, \frac{2}{3}\right)$. Consider a function $f: \mathbb{R} \to \mathbb{R}$ such that $$f(x) = inf\{\|x - y\|: y \in S\}$$ Draw the graph of $f(x)$ and hence find the set of points where f is not differentiable.	**CO3**

Q8	Using Lagrange's mean value theorem prove that $tan^{-1} x - tan^{-1} y < x - y$, where $x > y$.	**CO3**
Q9	Show that $log_e(1 + e^x) = log_e 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} + \cdots$ and hence deduce that $\frac{e^x}{1+e^x} = \frac{1}{2} + \frac{x}{4} - \frac{x^3}{48} + \cdots$ **OR** State and prove Taylor's theorem with Cauchy's form of remainder.	**CO4**
SECTION-C (This question carries 20 marks)		
Q 10	Let $f(x)$ be defined on $\mathbb{R}$ such that $\lvert f(x)\rvert \leq \lvert x\rvert \ \forall\, x \in \mathbb{R}$ and $f(x + y) = f(x) + f(y) \forall\, x, y \in \mathbb{R}$ then show that $f(x)$ is continuous on $\mathbb{R}$ and $f(x) = cx$.	**CO2**
Q 11	Consider a function $f: \mathbb{R} \to \mathbb{R}$ such that $\lvert f(x)\rvert \leq x^2$ for all $x \in \mathbb{R}$. Prove that $f(x)$ is differentiable at 0 by using Sandwich theorem. **OR** Prove that between any two roots of $e^x \sin x = 1$, there is at least one root of $e^x \cos x + 1 = 0$.	**CO3**