Name: **Enrolment No:**	UPES UNIVERSITY OF TOMORROW

UNIVERSITY OF PETROLEUM AND ENERGY STUDIES

End Semester Examination, May 2023

Course: Ring Theory & Linear Algebra II **Semester: VI**
Program: B. Sc. (Hons.) Maths **Time: 03 hrs.**
Course Code: MATH 3023 **Max. Marks: 100**

Instructions:

SECTION A
(5Qx4M=20Marks)

S. No.		Marks	CO
Q1	Suppose S denotes the set of polynomials in x that have no linear term i.e. $$S = \{a_n x^n + a_{n-1}x^{n-1} + \ldots\ldots + a_2 x^2 + a_0 \mid n \in \mathbb{Z}_{\geq 0}, a_i \in \mathbb{Z}\}$$ Is x^2 an irreducible element in S? Is x^2 prime? Justify.	**4**	**CO1**
Q2	A polynomial of degree n has at most n zeros in $\mathbb{Z}_n$. Prove or disapprove by suitable counterexample.	**4**	**CO1**
Q3	Is $\mathbb{Z}[\sqrt{-5}]$ a principal ideal domain? Justify.	**4**	**CO1**
Q4	Consider the set $S = span\{(a, b, c)\} \subset \mathbb{R}^3$, where a, b, c are in arithmetic progression. Find the orthogonal complement $S^{\perp}$ in $\mathbb{R}^3$ w.r.t. Euclidean inner product.	**4**	**CO2**
Q5	Does there exist a linear map $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ such that $T^2 + I = O$ (where I is identity and O is null matrix in $\mathbb{R}^3$)? Justify your answer.	**4**	**CO2**

SECTION B
(4Qx10M= 40 Marks)

S. No.		Marks	CO
Q 6	Consider the polynomial $p(x) = 2x^5 - 4x^3 - 3$ in the ring $\mathbb{R}[x]$. Is $p(x)$ irreducible over $\mathbb{Q}$? Defend your answer with sound mathematical reasoning.	**10**	**CO1**
Q7	Show that the element $1 + \sqrt{5}$ is irreducible in $\mathbb{Z}[\sqrt{5}]$.	**10**	**CO1**
Q8	Suppose W is invariant under $T: V \rightarrow V$. Show that W is invariant under $f(T)$ for any polynomial $f(t)$.	**10**	**CO2**

Q9	Consider a vector space V over $\mathbb{R}$ and $u, v \in V$. Derive Cauchy-Schwarz inequality $(\langle u, v\rangle)^2 \leq \|\|u\|\|^2 \|\|v\|\|^2$. **OR** State Cayley-Hamilton theorem and give its topological proof.	**10**	**CO3**
SECTION-C **(2Qx20M=40 Marks)**			
Q10	Let V be a vector space of polynomials over $\mathbb{R}$ of degree ≤ 2. Let ϕ_1, ϕ_2 and ϕ_3 be the linear functionals on V defined as $$\phi_1(f(t)) = \int_0^1 f(t)dt\,, \quad \phi_2(f(t)) = f'(1), \quad \phi_3(f(t)) = f(0)$$ Here $f(t) = a + bt + ct^2 \in V$ and $f'(t)$ denotes the derivative of $f(t)$. Find the basis $\{f_1(t), f_2(t), f_3(t)\}$ of V that is dual to $\{\phi_1, \phi_2, \phi_3\}$.	**20**	**CO2**
Q11	Consider the set $S = \{(3,1), (2,2)\}$ in the inner product space $\mathbb{R}^2$ equipped with the conventional Euclidean inner product. Normalize the vectors of S using Gram-Schmidt process. **OR** Obtain an orthonormal basis from the given basis $\left\{\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right\}$ in the vector space of all 2×2 real matrices i.e. $M_2(\mathbb{R})$ equipped with the inner product defined as $\langle A, B\rangle = tr(B^T A)$, where B^T is the transposed matrix B.	**20**	**CO3**