STUDY OF DYNAMICAL BEHAVIOR OF MATHEMATICAL MODELS IN BIOLOGY

A Thesis submitted to the
UPES

For the Award of
Doctor of Philosophy
in
Mathematics

By
Ravi Kiran Maddali

December 2023

SUPERVISOR(s)

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Dr. Sk. Sarif Hassan

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School of Advanced Engineering
UPES
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DECLARATION

I declare that the thesis entitled "**Study of Dynamical Behavior of Mathematical Models in Biology**" has been prepared by me under the guidance of Dr. Divya Ahluwalia, Associate Professor of Mathematics, UPES, Dehradun, Uttarakhand, India and Dr. Sk. Sarif Hassan, Assistant Professor & Head, Department of Mathematics, Pingla Thana Mahavidyalaya, P.O. Maligram, West Bengal, India. No part of this thesis has formed the basis for the award of any degree or fellowship previously.

2.3.24

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CERTIFICATE

I certify that **Ravi Kiran Maddali** has prepared his thesis entitled **"Study of Dynamical Behavior of Mathematical Models in Biology",** for the award of PhD degree of the UPES, under my guidance. He has carried out the work at the Department of Mathematics, Applied Science Cluster, UPES.

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ABSTRACT

Mathematical modeling is a significant tool for studying the behavior of biological systems, which are dynamical in nature. The parameters of the biological system undergo changes due to the dynamical nature and in due course of time, the system will become adaptive to these changes. The mathematical study of biological systems makes us understand several questions dealing with the growth of the biological organisms, interactions among them, stability of the system, etc., making us derive a complete picture of the system at a single instance. In the literature, one can find several mathematical models that are developed and successfully implemented in several areas of biology such as ecology and ecosystems, cell organization, evolution and diversity, growth and development, Genome expression, cancer modeling, etc.

In this thesis, the main emphasis was on the concept of prey-predator modeling. Prey-predator modeling is widely applied in the study of bio-systems and ecosystems. In these systems, the species always compete, grow and disappear during the process of seeking resources to maintain their very existence. Such models have a variety of applications in the study of consumer-resource, plant-herbivore, tumor cells-immune, susceptible-infectious-recovered, and parasite-host systems.

The study discusses the computational dynamics of the chaotic cancer model in three dimensions, Nicholson-Bailey models as well as Rosenzweig – Macarthur prey-predator models and investigates the dynamics of these systems and their variants using qualitative analysis.

In the study of nonlinear three-dimensional cancer model, the effect of $\alpha, \beta,$ and γ parameters on the behavior of the system involving nonlinear interactions among the tumor cells $x(t)$, healthy host cells $y(t)$, and effector immune cells $z(t)$ interacting in a single tumor cell compartment was investigated. The study reveals that the model exhibits chaotic behavior for a certain range of positive

parameters $\alpha, \beta,$ and γ thereby establishing that all three parameters are very sensitive to the system.

In the computational dynamics of Nicholson-Bailey models, a very popular population-dynamics model, the consequences, when the range of the system parameters is extended from positive real numbers to real numbers are studied. The existence of chaos, periodic solutions, and stable/unstable equilibriums in the system was investigated. In addition to the study of the classical model, the study further investigated the dynamics of the scaled and noisy models designed from the classical Nicholson-Bailey model. The bilateral symmetry in fixed points was obtained and several interesting results on the chaotic and periodic solutions of the models were observed.

The computational study of the Rosenzweig-Macarthur prey-predator model, one of the popular models in ecological dynamics, was conducted and the model was studied by considering the parameters as complex numbers instead of positive real numbers. Also, different functional responses from both prey and predator perspectives were considered and compared the results.

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CHAPTER 1. INTRODUCTION

1.1 Introduction

Mathematical biology is an applied branch of science for understanding, representing and modeling biological processes by employing the tools of applied mathematics. It makes use of theoretical analysis and mathematical models to get insight into the principles governing the structure, development, and behavior of the systems. Mathematical biosciences emerged as an interdisciplinary science with significant contributions in the allied fields of biological sciences and medicine from the experts of various disciplines like mathematics, physics, and life sciences.

The impact of mathematics on biological research is evident from the scientific literature available. Mathematics has influenced almost all the areas of research in biology like cell structure and function, ecology, ecosystems, genetics, marine biology, immune system, DNA analysis, organism function and disease, neurobiology, plant biology and agriculture, infectious diseases, cellular automata, and cancer modeling[1]. The mathematical study of the above systems is helping the biological community to gain sufficient understanding, which is further used in the areas of medicine, artificial healthcare aids, and decision making concerned with the systems.

Most of the biological systems are complex in nature. Most of the components in the biological systems exhibit multiple ways of interactions leaving no scope to define proper rules of possible interactions. Examples of complex systems include living cells, organisms, ecosystems, the human brain, etc. There exist several dependencies, competitions, relationships among the components of the system and components with the external environment, which are very difficult to comprehend. Most of these systems exhibit properties like nonlinearity, emergence, and adaptation. Modern scientists are relying on the dynamical systems approach to study complex systems.

In mathematical models, Linear, static and deterministic models are usually easier to handle than non-linear, dynamic, and stochastic models. Continuous-variate models appear to be easier to handle than the discrete variate models, due to the development of differential equations. However, in absence of analytical solutions, one needs to transform a continuous model into a discrete model and handle them numerically [2].

There is a large scope for computational techniques while dealing with discrete mathematical models.

1.2 Dynamical Systems

A system whose state changes with time over a state space based on a designated rule is a dynamical system. Mathematically, it can be defined as a state-space X, a set of times T, and a rule R that defines how the state evolves over time. Here $R: X \times T \rightarrow X$. Dynamical system involves the evolution of something over time. The variables describing the state of a system are state variables. A state-space comprises all possible state variables. The dynamical system's dimension is dependent on the count of the state variables. The state space can be continuous or discrete. The continuous finite-dimensional state space is known as phase space. The system whose state evolves through a continuous-time is a continuous dynamical system and the system whose state evolves through a discrete-time is a discrete dynamical system.

Dynamical systems study has significant applications in the fields of biology, physics, sociology, engineering, and economics. Dynamical systems remain as the central idea of chaos theory, dynamics of logistic maps, bifurcation analysis, and fractals.

Henri Poincaré founded the dynamical systems theory. He compiled several interesting results in his research monographs titled 'New methods of celestial mechanics' and 'Lectures on celestial mechanics' [3], [4]. He studied the behavior of solutions to the problem related to three-body motion based on the above results. He proposed a famous 'Recurrence theorem' stating "dynamical systems returns nearer to their initial state after a finite interval of time which may be sufficiently long". Aleksandr Lyapunov developed the modern theory of stability of dynamical systems. He developed several approximation methods, which help one to get insight into the stability of ODEs.

The evolutionary rule in a dynamical system is generally a relation defined in terms of differential or difference equation. To get the future state of the system, this relation is iterated by incrementing time step. This process is known as finding the solution of the system. The set of all future positions obtained upon solving the system under certain initial conditions is known as *trajectory*. Previously, the task of finding the trajectory also known as orbit used to be a very tedious task involving the application of several mathematical sophisticated techniques but these days this task is simplified by implementing various numerical techniques on computers. Obtaining the trajectory is sufficient for understanding most of the simple dynamical systems. The study of dynamical systems introduced new theories like Chaos and Bifurcation.

1.3 Chaos

The chaotic theory deals with complex systems whose behavior becomes quite unpredictable and appears more random. Such systems are more sensitive to even slight alterations in the system's initial values. The behavior of such systems can be predicted only up to a certain extent and after some time, these become unpredictable by exhibiting random behavior. The time up to which these systems can be accurately predicted is dependent on factors such as tolerable uncertainty in predicting, accurate measurement in the current state, and Lyapunov time. Understanding the chaotic behavior of the systems is very important for a better understanding and decision-making. The chaotic behavior is most common in real-world systems like weather and climate. It has numerous applications in the areas of computer science, Biology, Finance, Economics, Engineering, Physics, and political science.

1.4 Bifurcation

Henri Poincaré introduced the concept of bifurcation. The Bifurcation theory deals with the sudden qualitative changes in the system that occurred due to small changes in the system parameters known as bifurcation parameters. Bifurcation can be noticed in both continuous and discrete dynamical systems. Bifurcation is further classified as local and global. The deviations in local stability properties of equilibria and periodic orbits can be studied for the understanding of local bifurcations whereas the study of stability at equilibria cannot be helpful to completely understand global bifurcation in the systems where there is a collision of larger invariant sets among themselves or with the equilibria of the system.

1.5 Stability

The stability theory deals with the dynamical systems stability. Stability theory helps us to understand the behavior of solutions of differential equations with slight changes in their initial conditions. Dynamical systems study addresses the long-standing behavior of systems for which one has to study the trajectories and asymptotic properties. The behavior exhibited by periodic orbits and fixed points is to be studied. After understanding a particular orbit, the natural question is what happens to the behavior of the system if we bring a small change in the initial condition. Stability theory deals with such investigations. If a nearby orbit stays close to a given orbit indefinitely, the orbit is known as a stable orbit. If the nearby orbit converges to the given orbit, then the orbit is asymptotically stable. The orbit given is known as attracting.

1.6 Linearization

Most of the real systems are nonlinear in nature. One cannot precisely find the solutions of nonlinear systems. The linearization method gives us an insight into the system's behavior in the neighborhood of equilibria. This technique helps approximate nonlinear differential equations as a system of linear ODEs. In the stability analysis of autonomous systems, the Jacobian at hyperbolic equilibrium is constructed, its Eigenvalues are considered, and the equilibrium's nature is obtained. The linearization theorem also referred to as the Hartman-Grobman theorem is significant in dynamical systems study. As per this theorem, the system behavior at hyperbolic equilibrium is similar to the behavior of its linearization at this equilibrium qualitatively. The behavior around the equilibria of the system can be studied by the linearization technique [5].

1.7 Dynamical Systems in Biology

Most of the questions concerning the evolution and long-term stability of highly complex biological systems can be answered effectively by the dynamical systems approach. The dynamical systems theory is broadly applied in the areas of biology and medicine. Various mathematical tools are successfully implemented to study biological problems related to cancer research, population biology, epidemiology, immunology, competition among species, harvesting, host-parasite systems, etc.

Ecology is one of the significant areas where one finds the vigorous application of dynamical systems study. The ecosystem comprises living organisms and non-living components in their environment. There will be interactions among the living organisms as well as organisms with its environment. These interactions may range from simple to very complex. Ecosystems are influenced both by external components which are not controllable and internal factors which are controllable up to a certain extent. These systems are dynamic as they are subjected to various changes in terms of periodic disturbances and these systems always try to overcome these disturbances. The studies in ecology deal with population dynamics, fisheries, competitions among species, epidemics, group dynamics, food webs, and effects of climatic changes.

Several growth models like logistic, exponential, and structured population models were developed for studying the dynamics of the species population. All these models come under population ecology. The main objective of such studies is to understand how the populations are interacting with their environment and how their respective sizes vary with respect to space and time. Several studies under population ecology are concerned with the habitation and resource needed for individual species, their behaviors as groups, and their growth. The conditions for abundant growth or extinction are also an interesting part of the study.

In an ecosystem, one comes across several species competing for the same resources. Prey-predator models, host-pathogen models, host-parasitoid models, competition, and mutualism models study the dynamics of community ecology. Community ecology investigates how the populations interact with one another and how they share common resources.

For instance, it is a well-known fact that resources are not abundant in nature and the life on our plant is to survive on the available limited resources. Due to industrialization and urbanization, one can see that populations are growing in such a fashion that the available resources could not cater to their needs. So, a proper understanding of the system is required for maintaining a proper balance in the system which is possible using mathematical modeling.

1.8 Some Studies in Literature

Lotka and Volterra had developed their models using differential equations, which deal with predator-prey, and competition-like situations [6], [7]. These are considered as fundamental models of mathematical ecology. Several authors have adopted the concept of predator-prey modeling for studying tumors and the immune system [8]–[16]. Similarly, significant research contributions to mathematical models in biology using host-parasite or predator-prey modeling were available in the literature. Some researchers have studied a variety of problems and conducted the stability analysis thereby further investigating the bifurcations and exhibition of chaos in the respective systems of their studies [17]–[31].

1.9 Objectives of the Thesis

In the literature, one can find many well-developed mathematical models in the domains of infectious diseases, predator/prey or host/parasite systems, competitive species, and competition & harvesting. Researchers from various domains study, implement and extend these models while investigating problems of their domain. Biological species always aim for their existence and engage in the quest for resources. Based on the application, prey-predator models can be viewed in different forms such as consumer-resource, herbivore-plant, host-parasite, tumor cells-immune systems, and susceptibles-infectors, etc. As these processes involve in loose-win-like situations, these models can be effectively implemented outside the ecosystems as well.

The wide variety of applications of dynamical system study of biological systems provided motivation for the qualitative study of various mathematical models in biology. The current work aimed at investigating

(1) The computational dynamics of a chaotic cancer model in three dimensions.
(2) The computational dynamics of the Nicholson-Bailey model, its scaled and noisy models.
(3) To study Rosenzweig-Macarthur predator-prey models by considering the parameters as complex numbers.
(4) To investigate stability, periodic solutions, bifurcations, and chaotic behavior in the above models.

CHAPTER 2. REVIEW OF LITERATURE, MATHEMATICAL TOOLS AND METHODS.

2.1 Introduction

Mathematical modeling is a tool of great significance in the study of the dynamics of biological systems. Numerous mathematical models have been constructed and successfully implemented in various studies related to the domains of ecology, population dynamics, genetics, epidemics, and medicine. These models are mathematical equations, whose study helps us to understand and predict the underlying natural phenomena such as the behavior of organisms, growth of the tumor, and population changes over time. Most of the researchers in mathematical biology employ dynamical systems theory in their investigations.

Mathematical ecology deals with the interactions of organisms and their environment.

2.2 Models from Literature:

1. Lotka-Volterra model developed independently by Lotka and Volterra [6] [7]. It describes the interactions among predator-prey populations with the following assumptions:
   (1) Sufficient food is available at all times to the prey population.
   (2) The predator population survives on the prey population.
   (3) The rate changes of both populations are based on their sizes.
   (4) There are no environmental changes during the prey-predator interactions.
   (5) The predators have limitless appetite.

The model obtained is

$$\begin{cases} \frac{dx}{dt} = rx - axy \\ \frac{dy}{dt} = bxy - my \end{cases} \qquad \text{Eqn. (2.1)}$$

The variables and parameters in the model are x −prey density, y −predator density, r −intrinsic rate of increase of prey, a −rate coefficient of predation, b − reproduction rate of predators per 1 prey consumed, and m − rate of predator's mortality. The system was studied by linearizing the model equations and by finding the equilibrium point's local stability.

The following are conclusions obtained from the study of solutions of the model:

(1) The model was not accurate, as the competition among prey and predator populations was not addressed.
(2) Both the populations grow infinitely due to limitless resources.
(3) Predators' consumption rate is unlimited.
(4) Asymptotic stability was not found making the model behavior unnatural.

Several modifications were made to this Lotka-Volterra model making the model more realistic.

2. Nicholson and Bailey (1935) developed a model known as the Nicholson-Bailey model. This model describes the coupled host-parasitoid system dynamics. This model uses difference equations while describing the growth dynamics of host-parasite populations and resembles the Lotka-Volterra model, which was modeled using differential equations [17].
   The following assumptions were made:

   (1) All infected hosts would produce a new generation of parasites.
   (2) All uninfected hosts will continue to have their own offspring.

(3) The encounters between hosts and parasites are random.
(4) These encounters result in the infection of hosts.
(5) Each host would be infected once by a parasite.

The model equations are described in discrete time given by

$$\begin{cases} H_{t+1} = kH_t e^{-p_s P_t} \\ P_{t+1} = c_a H_t (1 - e^{-p_s P_t}) \end{cases} \qquad \text{Eqn. (2.2)}$$

Here, H_t −population size of hosts at time t, P_t −population size of parasitoids at time t, k −host reproductive rate, p_s − the parasitoid searching efficiency, and c_a − the average count of eggs laid on a single host body by a parasitoid.

The system exhibits oscillations moving from an unstable equilibrium and it has one unstable fixed point

$$(H^*, P^*) = \left(\frac{k\, ln(k)}{(k-1)p_s c_a}, \frac{\ln(k)}{p_s}\right) \qquad \text{Eqn. (2.3)}$$

The model concludes that the system exhibits unstable oscillations until all the individuals of the population die. This simple model is non-spatial and the populations are considered as mixed-well. But, in realistic scenarios, this may not be true.

By considering the species interactions and the dispersal of offspring as local, the following spatial Nicholson-Bailey model was proposed:

$$\begin{cases} H_{i,j}(t+1) = kH_{i,j}(t)\, e^{-p_s P_{i,j}(t)} \\ P_{i,j}(t+1) = c_a H_{i,j}(t)(1 - e^{-p_s P_{i,j}(t)}) \end{cases} \qquad \text{Eqn. (2.4)}$$

At each grid point (i, j), the dynamics are almost described by simple Nicholson-Bailey model but in addition, hosts and parasitoids will disperse immediately to neighboring sites.

The indefinite coexistence of hosts and parasitoids is possible as per this spatial model whereas they eventually die in a non-spatial model.

3. Hassell and May studied the interactions among hosts and parasites based on the random search [18].

The generalized model considered was

$$\begin{cases} M_{t+1} = FM_t f[N_t, M_t] \\ N_{t+1} = M_t - \frac{M_{t+1}}{F} \end{cases} \qquad \text{Eqn. (2.5)}$$

The model variables are M —the host population density in t and $t+1$ generations, N —the parasite population densities in t and $t+1$ generations. $F-$ rate increase in hosts.

The authors developed a few models, which are briefed below:

MODEL A: The authors developed a model by introducing the Nicholson-Bailey host-parasite model in the form of control [17], [31]. Here the concept of random search is included. Both host and parasite population densities have no impact on the searching efficiency. The assumptions were (i) each parasite searches in random (ii) The average area searched in lifetime effectively by a parasite constant. (iii) Enough eggs are available with the parasite for oviposition in all the hosts, which it encounters.

MODEL B: This was constructed on the grounds of findings of Holling[32]. The searching efficiency is related to the host density. As per the model the handling time, the time taken for resuming the search activity after encountering a host gradually decreases the available search time as more hosts are encountered.

MODEL C: This model depends on the observation by Hassel that parasite searching efficiency depends on the density of searching parasites [33]. A little modification was made to the Nicholson Bailey model, which completely altered the outcome of the host-parasite model. The old model was always unstable. The new model was found to be stable on an extensive range of conditions that depend on the host increasing rate and the amount of interface.

All these models are random search models based on Poisson distribution.

The authors pointed that even though random search is mathematically convenient, in realistic situation randomness if not a rule but an exception. They developed two more models based on non-random search. The authors found that the host and parasite density functional responses, and the response to the host distribution are the three basic parasite responses effecting the parasite searching behavior.

For all these models, the significant parameters effecting the stability are discussed with needful illustrations. It was observed that the stability is influenced by factors like parasites mutual interference, aggregation of parasites in unit areas consisting high density of host population, and spatial or temporal asynchrony.

The parameters affecting the levels of equilibrium of both the populations as well as those affecting the stability were investigated. From this study, the biological control characteristics were found to be a low handling time, high searching efficiency, and aggregation of parasites.

4. Beddington and Hammond studied a mathematical model governing the interactions of a herbivorous host parasitized by a primary parasite which while developing itself subject to parasitism by the secondary parasite [23].

The assumptions are:

(1) The sequence of events in which hosts are parasitized by primaries, which then become available for attack by secondaries.

(2) There is no overlap of generations in any of the species.

(3) The transition from one time period to the next occurs after emerging of the secondary parasites.

The model equations are:

$$\begin{cases} M_{t+1} = M_t F_1(M_t) F_2(N_t) \\ N_{t+1} = M_t[1 - F_2(N_t)] F_3(S_t) \\ S_{t+1} = M_t[1 - F_2(N_t)][1 - F_3(S_t)] \end{cases} \quad \text{Eqn. (2.6)}$$

where $N_t, P_t,$ and S_t denote the densities of the host, primary parasite, and secondary parasite. The function $F_1(M_t)$ defines the host rate of increase as it depends on its own density. $F_2(N_t)$ defines the proportional host survival as a function of primary density and $F_3(S_t)$ defines as a function of secondary density, the proportion of primary parasites that survive the attack by secondaries. The authors studied the stability at equilibria.

The authors derived the conditions for secondary parasite invasion of either a stable host-parasite system or an oscillatory one. They studied the global stability of the analysis. The important parameters of the interaction are identified and their effects on the feasibility and stability of the system are documented. The study revealed that certain combinations of the parameters permit both a host-primary and a host primary-secondary system to have a locally stable equilibrium. Two effects relevant to biological control are noted. It was established that when the host-primary system is stable, a secondary will always weaken effective control of the host and when the host-primary system displays oscillations introduction of a secondary may result in a stable three species equilibrium. A stable three species equilibrium will have a small range of parameter space compared to that of a two-species system.

5. Ruan and Xiao studied a prey-predator system with a non-monotonic functional response [25]. For understanding the global dynamics of the system, the authors performed global qualitative and bifurcation analysis. They

discussed various bifurcation phenomena like saddle-node, supercritical and subcritical Hopf, and homoclinic bifurcations.

The authors studied the prey-predator model wherein the group defense was exhibited by prey, developed by Freedman and Wolkowicz [34]. The model equations are given by

$$\begin{cases} \dot{x} = xg(x, K) - yp(x) \\ \dot{y} = y\big(-D + q(x)\big) \end{cases} \qquad \text{Eqn. (2.7)}$$

Here the system variables and parameters are x −prey density, y −predator density, $\{K, D\} > 0$ are the prey carrying capacity and predator death rate. In predator's absence, the prey population growth rate is given by the function $g(x, K)$. $p(x)$ denotes predator response function and $q(x)$ is the rate of conversion of prey to predator.

On the model, the global qualitative analysis was conducted. The bifurcation parameters were taken as the prey carrying capacity and predator death rate. The study discovered new types of bifurcations and discussed global qualitative analysis for the particular case.

6. Kar considered a fishery predator-prey model. By incorporating a time delay in harvesting, the author studied the selective harvesting of fishes above a certain size and age [26]. This problem has a relevance that is more practical from a commercial point of view. The fisherman retains the bigger fish caught and throws back the smaller fish into the water bodies. This type of technique requires adjustments in the size of net mesh so that the smaller fish caught will swim back through the mesh gaps. The author studied generalized gauss type prey-predator models separately with prey and predator harvestings.

The first model (predator harvesting) considered after introducing the delay term is

$$\begin{cases} \frac{dx}{dt} = x[g(x) - yp(x)] \\ \frac{dy}{dt} = y[-d + \alpha xp(x)] - qEy(t-\tau) \end{cases} \qquad \text{Eqn. (2.8)}$$

Here $g(x)$ is the prey's specific growth rate in the predator's absence, $p(x)$, α, and d are the response function, conversion factor, and the predator's death rate in the prey's absence respectively. $\tau \geq 0$ (delay) is the harvesting term, q catchability coefficient of predator species, qEy is catch rate function and E is effort function.

The second model (prey harvesting) considered after introducing the delay term is

$$\begin{cases} \frac{dx}{dt} = x[g(x) - yp(x)] - qEx(t-\tau) \\ \frac{dy}{dt} = y[-d + \alpha xp(x)] \end{cases} \qquad \text{Eqn. (2.9)}$$

The author performed the stability analysis of these models.

The study revealed that any kind of instability was not induced by the delay of all dimensions. In certain situations, the delay is harmless. It was also observed that the instability oscillation through Hopf bifurcation gets induced by a delay of certain dimensions. In addition, stability may switch in some situations. These types of investigations with delay terms were not conducted before this work. Simulations with chosen artificial data on the two models were performed to verify certain obtained mathematical results.

7. Kar studied the prey-predator model with functional response considered as Holling type II [27]. The model incorporates a prey refuge with the main motto to conduct a rigorous mathematical investigation of the system model and to

provide qualitative results from the biological perspective. The study analyses a Lotka-Volterra prey-predator model with functional response considered as Michaelis-Menten type.

The assumptions are:

(1) The population density of prey is resource-limited.
(2) With the increase of the prey population, the functional response of each predator to the prey approaches a constant.
(3) A certain percentage of prey is protected from predation by a spatial refuge.

The model equations are

$$\begin{cases} \frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right) - \frac{\beta yx}{1+ax} \\ \frac{dy}{dt} = -\gamma y + \frac{c\beta yx}{1+ax} \end{cases} \qquad \text{Eqn. (2.10)}$$

The variables and parameters of the model are $x -$ population density of prey at time t, $y -$population density of predators at time t, $\{r, K, \gamma, \beta, a, c\} > 0$, $r -$ intrinsic rate of growth, $K -$prey's carrying capacity, $\gamma -$ predator's death rate, $\frac{\beta}{a}$ is the maximum number of prey that can be eaten by each predator in unit time, and c is the conversion factor denoting the number of newly born predators for each captured prey.

The study concludes that incorporating a refuse makes the model more realistic. A refuse is considered significant for pest's biological control. The prey densities will raise by an increase in the amount of refuse leading to an outbreak in the population. Conditions were derived for the existence of equilibria, their stability. The criteria for persistence were obtained. The results were verified by through numerical simulations.

8. Xu et al investigated a predator-prey model with stage structure for predator [36]. It is considered that the predator population may be classified as matures and immature. The maturity age was presented by a delay in time. Predators

who are still immature cannot prey. At non-negative equilibria, sufficient conditions for stability are derived.

The model equations of the study are

$$\begin{cases} \dot{x(t)} = x(t)(r_1 - a_{11}x(t) - a_{12}y_2(t) \\ \dot{y_1(t)} = \alpha x(t)y_2(t) - \gamma_c y_1(t) - \alpha e^{-\gamma_c t}x(t-\tau)y_2(t-\tau) \\ \dot{y_2(t)} = \alpha e^{-\gamma_c t}x(t-\tau)y_2(t-\tau) - r_2 y_2(t) - a_{22}y_2^2(t) \end{cases} \quad \text{Eqn. (2.11)}$$

The variables and parameters of the system are $x(t)$ −density of prey, $y_1(t)$ −density of immature predator, $y_2(t)-$ density of mature predator, $\{a_{11}, a_{12}, a_{22}, r_1, r_2, \alpha$ and $\gamma_c\} > 0$ and $\tau \geq 0$.

The model assumptions are:

(1) The prey species growth is of Lotka-Volterra nature, $r_{1,}$ and a_{11} are the intrinsic growth and intra-specific competition rates respectively.

(2) a_{12} is the maturing predator's capturing rate, $\frac{\alpha}{a_{12}}$ is the conversion rate of nutrients into reproduction of matured predators; a_{22} is the rate of death of the matured predators.

(3) There is proportionality between the immature population's death rate and the existing immature population. γ_c is the constant of proportionality.

The authors studied the positivity and boundedness of solutions, global stability at non-negative equilibria, local asymptotic stability and proposed various mathematical results. The results were verified through numerical simulations.

9. Das et al studied the bio-economic harvesting of a predator-prey fishery [30]. The release of toxicants by both species affects one another. The bio-economic studies deal with effective exploitation management of renewable resources.

The assumptions of the study are:

(1) The release of toxicants by sources such as industrial waste has an impact on the growth of individual species.

(2) The intoxicants have different effects on both populations.

(3) The predators alone influence prey reproduction. The amount of prey caught influences the predator reproduction.

(4) When there are no predators, the prey density increases with a relative rate r_1 and in case of non-existence of prey, predator population exponentially falls with a relative rate r_2.

The model equations are given by

$$\begin{cases} \frac{dx_1}{dt} = rx_1\left(1 - \frac{x_1}{L_1}\right) - \alpha x_1 x_2 \\ \frac{dx_2}{dt} = -dx_2 + \beta x_1 x_2 \end{cases} \qquad \text{Eqn. (2.12)}$$

The variables and parameters of the model are x_1 −size of prey, x_2 −size of predators at time *t*, r − maximum specific rate of growth of prey species, d − relative rate of death of predators prey's absence, and L_1 − environmental carrying capacity of prey population.

The steady state and their stability, global stability and bio-economic equilibrium were discussed.

The optimal harvest policy was determined. The existence of limit cycles was discussed. The bio-economic equilibria were found to have occurred at the intersections of the lines of biological equilibrium and zero profit. Through optimal harvesting policy, it was established that revenue can be maximized by zero discounting and that the economic rent is completely degenerated by an infinite discount rate.

10. Das et al studied the problem of nonselective harvesting with an appropriate catch-rate function [19]. Here, both the populations are assumed to be following the logistic growth law. The authors considered the functional response of predator to prey density such that individual predator's functional response to

the density of prey moves towards a constant value with the increase of population of prey.

The local stability of equilibria of the fishery system is discussed with necessary illustrations. The stability of the interior steady state is obtained and it was established that the system is bounded.

The model considered after assuming the combined harvesting effort E is

$$\begin{cases} \frac{dx_1}{dt} = r_1 x_1 \left(1 - \frac{x_1}{L_1}\right) - \frac{m x_1 x_2}{a + x_1} - \frac{c_1 E x_1}{l_1 E + l_2 x_1} \\ \frac{dx_2}{dt} = r_2 x_2 \left(1 - \frac{x_2}{L_2}\right) - \frac{m \beta x_1 x_2}{a + x_1} - \frac{c_2 E x_2}{l_3 E + l_4 x_2} \end{cases} \quad \text{Eqn. (2.13)}$$

The variables and parameters of the system are x_1 −size of prey population,x_2 −size of predator population, L_1 −environmental carrying capacity of prey, L_2 − environmental carrying capacity predators, β − conversion factor, m − maximum relative increase of predation, r_1 −intrinsic growth rate of prey, r_2 − intrinsic growth rate of predator; $E(t)$ −effort function; $\{c_1, c_2\}$ are the coefficients of catchability of both the populations.

The equilibrium points are found. Boundedness, local stability, and global stability were studied. The bio-economic equilibrium and optimal harvesting policy were studied. It was established that revenue can be maximized by zero discounting. Also, observed that with an infinite discount rate, the economic rent has totally degenerated.

11. Kar and Chakraborthy considered the predator-prey type fishery model [29]. For the prey population, a partial closure is considered. The dynamics of the system are studied by obtaining steady states. By formulating a policy for optimal harvesting, the solutions are obtained by Pontryagin's maximal principle. Numerical examples were illustrated to support the obtained results.

The assumptions of the model discussed are:

(1) The prey is harvested continuously.

(2) Due to the non-harvesting of predators, the harvesting has no impact on the predator.

(3) Both the populations fight for a common resource.

(4) The growth of prey is logistic.

The model equations are given by

$$\begin{cases} \frac{dx_1}{dt} = rx_1\left(1 - \frac{x_1}{k}\right) - \frac{\alpha x_1 y_1}{M_c + x_1} - h(t) \\ \frac{dy_1}{dt} = -dy_1 + \frac{\beta\alpha x_1 y_1}{M_c + x_1} - \gamma {y_1}^2 \end{cases} \qquad \text{Eqn. (2.14)}$$

The variables and parameters of the system are $x_1 -$ size of prey population, $y_1 -$ size of predator population, $r -$prey's growth rate, $K -$carrying capacity; $\alpha -$ conversion factor denoting maximal relative increase of predation, $M_c -$ Michaelis-Menton constant, $h(t) -$ harvesting at the time t, $d -$ predator's death rate and $\beta -$ conversion factor.

The uniform boundedness of the system is studied and theorems were established. Existence, as well as stability of interior equilibrium points, are studied. Bifurcation analysis was carried out and various numerical examples were illustrated by considering hypothetical data.

12. Rebaza studied the system with prey refuge and continuous threshold prey harvesting [38]. The study focused on how the refuge and harvesting have their impact on the ecosystem. The stability of equilibria and periodic solutions were discussed. Both hypothetical and computational techniques were used in investigations. In addition, bifurcations of various natures were studied.

The assumptions in this study are as follows:

(1) The basic interactions of prey and predator are governed by Michaelis-Menten functional response.

(2) The external agent is used to harvest prey.

(3) Prey refuge is included in the system to counterbalance predation.

The model equations are given by

$$\begin{cases} \dot{x_1} = x_1(1-x_1) - \frac{a_p(1-m)x_1y_1}{1+c(1-m)x_1} - H(x) \\ \dot{y}_1 = y_1(-d + \frac{b_p(1-m)x_1}{1+c(1-m)x_1}) \end{cases} \qquad \text{Eqn. (2.15)}$$

The variables and parameters of the system are x_1 −size of prey population, $y_1 -$ size of predator populations, the constants $\{a, b, c, d, h, m, T\} > 0$, $a_p -$ rate of capture of the prey, $b_p -$ rate of conversion of prey , $d -$ natural rate of death of the predator, $H(x) -$ harvesting function with continuous threshold policy on the prey.

The authors studied the boundedness of solutions, equilibria and stability properties, bio-economic equilibrium, and bifurcations.

The study concludes that compared to constant or linear harvesting, continuous threshold harvesting on the prey is a better policy. The effects of harvesting and refuge on the system are discussed. The obtained periodic solutions are found to be unstable.

13. Qureshi et al conducted the qualitative study of the Nicholson-Bailey host-parasite model and discussed the unique positive equilibrium point's local asymptotic stability [41].

The model assumptions are:

(1) encounters between hosts and parasites are random.

(2) per parasite search area and the root of the parasitoid density are in inverse proportion to each other.

(3) A host can escape the parasitism with a probability approximated by $e^{-a\sqrt{y_n}}$.

The discrete model equations are given by

$$\begin{cases} x_{n+1} = Rx_n e^{-a\sqrt{y_n}} \\ y_{n+1} = x_n(1 - e^{-a\sqrt{y_n}} \end{cases} \qquad \text{Eqn. (2.16)}$$

where x_n, y_n represents the n^{th} year population densities of hosts and parasites; R is the offspring count of a host (not parasitized) surviving to the next year.

The local asymptotic stability is observed for the unique positive equilibrium point of the system using the linearization method. Numerical examples were provided in support of the established theorems.

14. Gao and Jin studied a more realistic three-stage-structure prey-predator model [42]. Introducing a time delay, the authors investigated a system with "three-stage-structure" for predator. The functional response was Beddington-DeAngelis form. The model is

$$\begin{cases} \frac{dX_1}{dt} = \frac{d(t)x(t-\tau_2(t))X_2(t-\tau_2(t))}{\alpha(t)+\beta(t)x\left(t-\tau_2(t)\right)+\gamma(t)X_2(t-\tau_2(t))} - (d_1(t)+r_{im}(t))X_1(t) \\ \frac{dX_2}{dt} = r_{im}(t)X_1(t) - d_2(t)X_2(t) - r_{mo}(t)X_2(t) - e_2(t)X_2^2(t) \\ \frac{dX_3}{dt} = r_{mo}(t)X_2(t) - d_3(t)X_3(t) \\ \frac{dx}{dt} = x(t)\left(b(t) - a(t)X_1(t-\tau_1(t) - \frac{c(t)X_2(t)}{\alpha(t)+\beta(t)x(t)+\gamma(t)X_3(t)})\right) \end{cases}$$

Eqn. (2.17)

with initial conditions

$X_i(\theta) = \varphi_i(\theta), i = 1,2,3,\dots; y(\theta) = \varphi_4(\theta) > 0 \; for \; \theta \in [-\tau, 0]$, and

$X_i(0) > 0, i = 1,2,3 \dots, y(0) > 0$.

The variables and parameters in the model are X_1 −density of the immature predator at time t, $X_2 -$ density of mature predator at time t, $X_3 -$ density of old age predator at time t, $x(t) -$ prey density at time t, $\tau -$ delay due to prey densities, $\tau_2(t) -$ predator gestation-delay at time t, d_1 −death rate of immature predator; $d_2 -$ death rate of matured predator and d_3 −death rate of old age predator. $r_{im} -$ the rate of transformation from immature to mature predators and $r_{mo} -$ the rate of transformation mature to old age predators respectively.

Positive periodic solutions are examined. The authors established that the delay term has no negative impact on the positive periodic solution.

15. Zhang and Zhang investigated a bio-economic model with stochastic fluctuations [43],. A transition from stable to unstable and back again to stable was observed in internal equilibrium with the increase in time delay. Hopf bifurcations were determined.

The authors considered a time delay "Leslie-Gower prey-predator" system and introduced discrete delay and the transformed system is given by

$$\begin{cases} \dot{x(t)} = r_1 x(t) - \left(1 - \frac{x(t-\tau)}{K}\right) - \beta x(t) y(t) \\ \dot{y(t)} = r_2 y(t) - \left(1 - \frac{y(t)}{\gamma x(t)}\right) \end{cases} \qquad \text{Eqn. (2.18)}$$

The notations in the model are $x -$ density of prey population, $y -$ density of predator population, $r_1, r_2 -$ intrinsic growth rates of prey and predators, $K -$ carrying capacity of prey, $\gamma x(t) -$ prey-dependent carrying capacity of the predator and $\beta x(t)$ −predator functional response to prey.

The authors discussed the stability of equilibria and Hopf bifurcations. A time delay model in the fluctuating environment was proposed and numerical simulations were presented.

16. Khan and Qureshi conducted the qualitative analysis on the modified Nicholson-Bailey host-parasite model [44].

The model assumes that

(1) When the parasitoids are absent, the host exhibits bounded dynamics.

(2) e^{-ay_n} is the probability for the host to escape parasitism, a is the constant of proportionality constant and y_n is the density of parasitoid population.

(3) The probability for the host to become infected is $1 - e^{-ay_n}$.

The Nicholson-Bailey modified model is given by

$$\begin{cases} x_{n+1} = \frac{bx_n e^{-ay_n}}{1+dx_n} \\ y_{n+1} = cx_n(1 - e^{-ay_n}) \end{cases} \qquad \text{Eqn. (2.19)}$$

where $a, b, c, d,$ and the initial values $x_0, y_0 \in \mathbb{R}^+$, $x_n -$ density of prey population, $y_n -$ density of predator population, $a -$ each parasitoid attacks the hosts found in 'a' units of area, and $b -$ number of offsprings of an un-parasitized host surviving to the next year.

The authors studied the boundedness, existence and uniqueness of positive equilibrium. Its locally asymptotic and global stabilities were also investigated. For positive solutions, the rate of convergence was obtained.

17. Chaudhuri and Hassan studied the coupled-prey predator dynamics with mature-immature stage structure [21]. The results were compared with those of the existing model of Xin-an Zhang et al [24]. The model is realistic with parameters assumed as positive [24]. The authors studied the model with complex parameters.

The discrete model equations studied are

$$\begin{cases} x_{t+1} = x_t + (ay_t - bx_t - cx_t^2 - dx_t z_t)dt \\ y_{t+1} = y_t + (x_t - y_t)dt \\ z_{t+1} = z_t + \left(z_t(-e + x_t - z_t)\right)dt \end{cases} \qquad \text{Eqn. (2.20)}$$

where a, b, c, d, e are complex numbers and $dt = 0.0005$ is the delay term. Here $x(t)$ −population size of immature prey at time t, $y(t)$ −population size of mature prey at time t, $z(t)$ −population size of predator at time t.

The authors investigated the system using complex parameters and observed the dynamics as more complex compared to the classical realistic model.

The authors modified the model to observe the permanence of both the populations and studied the stability.

The discrete model equations for the modified model are

$$\begin{cases} x_{t+1} = x_t + (ax_t - by_t - cx_t^2 - dx_t z_t)dt \\ y_{t+1} = y_t + (y_t(x_t - y_t))dt \\ z_{t+1} = z_t + \big(z_t(-e + x_t - z_t)\big)dt \end{cases} \qquad \text{Eqn. (2.21)}$$

where $a, b, c, d \; and \; e$ are complex and dt is the delay term in discretizing the system.

The modified system has two additional fixed points including all the points from the previous model. One kind is that only immature population will be permanent and the other kind would be only immature prey and predator will be permanent.

18. Atabaigi and Akrami considered a two-parameter family of discrete models, consisting of coupled nonlinear difference equations describing a host-parasite interaction [45]. The stability and bifurcation analysis was done on the model whose equations are given by

$$\begin{cases} H_{n+1} = rH_n(1 - H_n)\, e^{-p_s P_n} \\ P_{t+1} = H_n(1 - e^{-p_s n}) \end{cases} \qquad \text{Eqn. (2.22)}$$

The variables and parameters of the system are $H_n - n^{\text{th}}$ generation host population, $P_n - n^{\text{th}}$ generation predator population, $r -$ host's intrinsic growth rate, $p_s -$ parasite's searching efficiency.

If the parasites vanish in the system, then it becomes the discrete logistic type model.

The model assumptions are

(1) In the parasite's absence, the host equation becomes a logistic model.

(2) The host's presence alone makes the parasites grow and the effect of parasites is the reduction of the size of the host population.

(3) The interaction is through the independent and random search by the parasite with constant and searching efficiency.

The authors studied the local dynamics of the model. The fixed points obtained are $(0,0)\ and\ \left(\frac{r-1}{r}, 0\right)$. if $r \in (0,1)$, the origin is asymptotically stable, and if $r > 1$, it is unstable. For $r \in (1,3)$ and $p_s < \frac{r}{r+1}$, the point $\left(\frac{r-1}{r}, 0\right)$ is asymptotically stable and when $r > 3$ or $p_s > \frac{r}{r+1}$, it is unstable.

The authors also investigated the presence of an unstable non-hyperbolic fixed point $(0,0)$ if $r = 1$.

The authors derived the sufficient conditions for Neimark-Sacker bifurcation.

The study of stability and bifurcation agrees with the biological facts. It was established that the parasite cannot grow in the host's absence and the interaction between host and parasite is given by a Poisson distribution's zero term The numerical simulations carried outmatched the theoretical results.

19. Hassan studied the three species model of Previte and Hoffman developed in 2013 [20], [29]. The author studied the behavior of the three species model

by slightly introducing immigration into the three populations[22]. The dynamics of the system with and without immigration were compared.

The model of Previte and Hoffman is given by

$$\begin{cases} \dot{x} = x(1 - y - z - bx) \\ \dot{y} = y(-c + x) \\ \dot{z} = z(-e + fx + gy - \beta z) \end{cases} \qquad \text{Eqn. (2.23)}$$

The variables and parameters of the system are x −prey density, y −predator density, z −omnivore density. All the other parameters are considered positive.

The author added immigration factors $p(x), q(y) and\ r(z)$ into prey, predator, and omnivore populations respectively and the following new model equations were developed.

$$\begin{cases} \dot{x} = x(1 - y - z - bx) + p(x) \\ \dot{y} = y(-c + x) + q(y) \\ \dot{z} = z(-e + fx + gy - \beta z) + r(z) \end{cases} \qquad \text{Eqn. (2.24)}$$

The qualitative study of the dynamics of the system is conducted and it was observed that the dynamics remained unchanged. This shows that the system is robust even after adding factors of immigration.

20. Bischi et al proposed a model related to fisheries [46]. Here harvesting process is conducted by fishermen who can harvest any one of the two species at a time. Based on individual agents' strategy, at each period of time, the fishermen are classified as two classes. Following an evolutionary mechanism, the agents switch between strategies to maximize respective profits. This study compares the economic consequences of self-regulating fishery and other regulatory policies.

The bio-economic model considered by the authors is given by

$$\begin{cases} \dot{T}_1 = T_1 G_1(T_1, T_2) - H_1(T_1, T_2) \\ \dot{T}_2 = T_2 G_2(T_1, T_2) - H_2(T_1, T_2) \end{cases} \qquad \text{Eqn. (2.25)}$$

where $\dot{T_i}$, $i = 1,2$, represents the biomass time derivatives, G_i is the growth functions, and H_i represents the species instantaneous harvesting.

For growth functions, the authors used the Rosenzweig-MacArthur prey-predator model. The Holling type-II functional response was considered.

The authors studied dynamic fishery with unrestricted and restricted harvesting and discussed bifurcations; proposed various analytical results. The study observes that there is a reasonable tradeoff between conservation of resources and maximization of profits with myopic, evolutionary self-regulation.

21. Yang and Li studied the diffusive prey-predator system with modified Leslie-Gower and Holling-type III schemes [48]. The stability of equilibrium is studied. For unique positive equilibrium, using the Lyapunov function, the global asymptotical stability conditions were obtained.

The system equations are given by

$$\begin{cases} \frac{dx}{dt} = x\left(a_1 - bx - \frac{c_1 xy}{x^2+k_1}\right) \\ \frac{dy}{dt} = y\left(a_2 - \frac{c_2 y}{x+k_2}\right) \end{cases} \qquad \text{Eqn. (2.26)}$$

The variables and parameters of the system are x −prey density, y −predator density, $\{a_1, a_2, b, c_1, c_2, k_1, k_2\} > 0$; a_1 − prey growth rate, a_2 −predator growth rate, b − individual species strength of competition among x, c_1 − maximum per capita reduction rate of x due to y, k_1 −extent of environmental protection of prey, k_2 − extent of environmental protection of predators.

The stability and permanence are studied. Various mathematical results are derived.

Several authors also successfully applied prey-predator modeling for the study of various problems related to multiple species modeling in harvesting, competition among two species with the third species as predator, periodicity of dynamical systems on time scales, discrete semi-ratio-dependent models with a functional response of the periodic solutions, dynamic behavior of stratum of plant-herbivore, bio-economic models in fisheries, three-dimensional prey-predator systems with prey disease, a diffusive predator-prey model with an epidemic in the prey species [20][24] [28][35] [37] [40] [46] [49] [50] [51] [52] [53].

2.3 Dynamical System

A system is all the activities united by some form of regular interactions. A dynamical system is a system that evolves over time [5], [58], [59]. It has a state for every point in time, which is subjected to a rule called the rule of evolution. The rule of evolution determines the next state of the system based on the preceding states. The dynamics of the system describes whether the system is going to settle down to an equilibrium state; becomes fixed into oscillating cycles or fluctuates chaotically. In certain cases, bifurcation occurs where small perturbations bring state changes in the system.

In discrete-time scenario, the state of the system evolves in integer time steps $= 0,1,2, \ldots$. Such system is referred as a discrete dynamical system. In a continuous dynamical system case, the system state evolves through continuous time. The state $x(t)$ can be thought of as a point moving smoothly in state space. The evolution rule decides the movement of $x(t)$ by giving its velocity. Here, starting with the initial state $x(0)$, the trajectory $x(t)$ is a curve through the state space. Continuous and discrete dynamical systems are modeled by differential and difference equations respectively.

2.4 Qualitative Study of Dynamical Systems

Dynamical systems are usually modeled as differential equations and there are relatively few equations possessing explicit solutions. So, most of the researchers use qualitative methods to study the behavior of these systems.

H. Poincaré and A.M. Lyapunov developed the basic qualitative theory of differential equations. Poincaré studied the solutions of systems of differential equations in an appropriate space by using geometric methods. The results so obtained helped him to develop a general theory on solution behavior. He studied numerous fundamental problems on the effect of parameters on the system's behavior.

Lyapunov founded the modern theory of stability of motion. His main concern was to study the dynamics of the solutions around equilibria.

2.5 Discrete Linear Models

The time discrete models are observed only at discrete times t_i, for $i = 0,1,2,3 \ldots$ [60], [61]. A generalized two dimensional discrete linear model is represented as

$$x_{n+1} = a_{11}x_n + a_{12}y_n$$

$$y_{n+1} = a_{21}x_n + a_{22}y_n$$

Eqn. (2.27)

Eqn (2.27) can be written as

$$\begin{pmatrix} x \\ y \end{pmatrix}_{n+1} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}_n \quad \text{Eqn. (2.28)}$$

The **stationary state** for a $x_{n+1} = \emptyset(x_n)$ is a $\bar{x}$, $\ni$ $\bar{x} = \emptyset(\bar{x})$. Therefore, $(\bar{x}, \bar{y}) = (0,0)$ is always a stationary state.

Consider the system

$$X_{n+1} = AX_n \quad \text{Eqn. (2.29)}$$

If X_0 is the initial condition. The solution of Eqn. (2.29) is

$$X_n = A^n X_0, n = 0,1,2, \dots .$$

We have $A^n X = \lambda^n X$ and $X_n = \lambda^n X_0$ satisfies the given system Eqn. (2.29), where λ is an eigenvalue and X is its corresponding eigenvector.

For matrix A, the spectral radius is $\rho(A) = \max(|\lambda|: \lambda\ is\ eigenvalue\ of\ A)$

For real eigenvalues, we have the following cases.

Case 1*a*: $0 < \lambda_1 < \lambda_2 < 1 \Rightarrow (0,0)$ *is a stable* node.

Case 1*b*: $0 < \lambda_1 = \lambda_2 < 1 \Rightarrow (0,0)$ *is a stable* one tangent node.

Case 2: $1 < \lambda_1 < \lambda_2 \Rightarrow (0,0)$ *is an unstable* node.

Case 3: $-1 < \lambda_1 < 0 < \lambda_2 < 1 \Rightarrow (0,0)$ *is a stable* node with reflection.

Case 4: $\lambda_1 < -1 < 1 < \lambda_2 \Rightarrow (0,0)$ *is an unstable* node with reflection.

Case 5: $0 < \lambda_1 < 1 < \lambda_2 \Rightarrow (0,0)$ *is a saddle point*.

Case 6: $-1 < \lambda_1 < 0 < 1 < \lambda_2 \Rightarrow (0,0)$ *is a saddle point* with reflection.

For complex eigenvalues, the following cases will arise:

Case 7: $\alpha^2 + \beta^2 = 1 \Rightarrow (0,0)$ *is a* center.

Case 8: $\alpha^2 + \beta^2 > 1 \Rightarrow (0,0)$ *is a*n unstable spiral.

Case 9: $\alpha^2 + \beta^2 < 1 \Rightarrow (0,0)$ *is a stable spiral*

A detailed discussion on the solutions along with their pictorial representations and certain standard theorems can be obtained in [61].

2.6 Nonlinear Discrete Models

The general form of one dimensional nonlinear difference equation of first order is

$$X_{n+1} = f(X_n) \qquad \text{Eqn (2.30)}$$

Definitions

(a) For the system Eqn. (2.30), if $\bar{x} = f(\bar{x})$, then $\bar{x}$ is called as a stationary point.

(b) The stationary point $\bar{x}$ of Eqn (2.30) is locally asymptotically stable $\exists$ a *nbd* U of $\bar{x} \ni$ for each starting value $x_0 \in U$, we get $lim_{n\to\infty} x_n = \bar{x}$.

(c) $\bar{x}$ is unstable , if it is not locally asymptotically stable.

Let f be differentiable. A stationary point $\bar{x}$ of $X_{n+1} = f(X_n)$ is locally asymptotically stable if $|f'(\bar{x})| < 1$. if $|f'(\bar{x})| > 1$, it is unstable. These are sufficient conditions but not necessary.

For the following discrete dynamical system, the stability of the system can be obtained by Jacobian matrix at stationary point $(\bar{x}, \bar{y})$.

$$\begin{cases} X_{n+1} = f(X_n, Y_n) \\ Y_{n+1} = g(X_n, Y_n) \end{cases} \qquad \text{Eqn. (2.31)}$$

where the stationary states $\bar{x}$ and $\bar{y}$ satisfy

$$\bar{x} = f(\bar{x}, \bar{y})$$

$$\bar{y} = g(\bar{x}, y)$$

Let $(\bar{x}, \bar{y})$ be a stationary state of the system Eqn. (2.31)

Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ be the Jacobian at the point $(\bar{x}, \bar{y})$ with eigenvalues λ_1, λ_2.

$(\bar{x}, \bar{y})$ is locally stable if $|\lambda_{1,2}| < 1$ and unstable if $|\lambda_j| > 1$ for one $j \in \{1,2\}$.

2.7 Routh-Hurwitz theorem

The signature of the real part of the eigenvalues decides the stability of equilibrium point. To find the solution's stability, it is enough to check whether the signs of real part of eigenvalues are negative or not. The method to check whether all the polynomial roots have their real parts as negative is discovered by Edward John Routh and Adolf Hurwitz independently formulated as Routh-Hurwitz theorem, which states that real parts of all the roots of the polynomial

$$\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n = 0 \quad \text{Eqn. (2.32)}$$

are negative if every coefficient a_i is positive and if every upper-left determinant $\Delta_i (i = 1,2, \ldots n)$ of the Hurwitz matrix **H** is also positive.

2.8 Bifurcation Analysis

If the dynamical system exhibits sudden qualitative changes in its behavior for small changes made in the parameter values, bifurcation occurs [62]. Both continuous and discrete systems can exhibit bifurcations. The concept of bifurcation was introduced by Henri Poincare. Bifurcation analysis speaks about structural stability. In a dynamical system, if small perturbations in parameters have no impact on the flow quality in the phase space, the system is structurally stable otherwise not.

2.8.1 Types of Bifurcation

The following are the classifications in bifurcation [62].

(a) Local bifurcation: Whenever the stability of the fixed point changes due to a change in parameter values, a local bifurcation occurs. At the bifurcation point, the equilibrium is non-hyperbolic. As parameters cross critical thresholds, this bifurcation can be studied from the variations in local stability of equilibria, periodic orbits.

(b) Global bifurcation: This occur when there is a collision between large invariant sets with equilibria. Here the changes in phase space are not limited to a small neighborhood. These cannot be studied completely with the help of stability analysis at fixed points.

2.8.2 Examples of Local Bifurcations

(a) Saddle Node Bifurcation: This is also called as fold bifurcation. It is related to a continuous dynamical system. The fixed points annihilate each other by colliding with one another.

(b) Transcritical Bifurcation: Here for all parameter values, the fixed point never ceases to exist. However, whenever a parameter value is changed, the fixed point's stability gets interchanged with another fixed point. When collided with one another, the stable fixed points become unstable and vice versa.

(c)Pitchfork Bifurcation: It is a form of local bifurcation. Here the transition from one fixed point to 3 fixed points takes place. Usually, this occurs in systems with reflection symmetry. This is further divided into supercritical and subcritical bifurcation.

(d) Hopf Bifurcation: In this type of bifurcation, when the equilibrium changes its stability through a pair of eigenvalues that are purely imaginary, a limit cycle arises from the equilibrium. This generally occurs in a two-dimensional system. This is again classified as supercritical and subcritical. Supercritical bifurcation occurs when the first Lyapunov coefficient obtained from Hopf bifurcation's normal form is negative. In the other case, the Hopf bifurcation is subcritical. An Orbitally stable and unstable limit cycle is obtained in supercritical and subcritical cases respectively.

2.8.3 Types of Global Bifurcation

The following are the types of global bifurcation [63][64].

(a) Homoclinic bifurcation: This occurs when there is a collision between saddle point and a periodic orbit. The period of the periodic orbit will grow to infinity at the point of bifurcation and becomes a homoclinic orbit. The periodic orbit ceases to exist after bifurcation.

(b) Heteroclinic bifurcation: This occurs when a steady point's unstable manifold becomes another steady point's stable manifold. The system is said to be having a heteroclinic connection where the two steady points are actually connected. When this steady point connection is broken, the heteroclinic bifurcation is said to occur.

(c)Infinite-period bifurcation: On a limit cycle, if two fixed points appear, then this bifurcation occurs. The period tends to infinity as the parameters limit approaches to a certain critical value. This bifurcation occurs at critical value.

2.9 Chaos

Chaos is a mathematical theory focused on dynamical systems which are highly sensitive to the initial conditions [65], [66]. This phenomenon was introduced by Edward Lorenz [67].

(1) It deals with non-linear, hard-to-control phenomena in our world.

(2) It helps to understand complicated behavior or natural occurrences while studying large and complex systems (systems with many components which make them difficult to understand)

The Climatic system, the pattern of bird's migration, behavior of boiling water, etc. are examples of complex systems.

2.9.1 Principles of Chaos theory

The butterfly effect: This speaks about the general idea that smaller causes may have larger effects. As per this effect, a tornado on the Japanese coast can occur due to the flapping of wings by a small butterfly in Mexico. Mathematically, there is a greater significance to initial conditions as they have a major potential to affect the outcomes. Small things at the beginning may lead to something major in the end.

Unpredictability: In the highly complex dynamical situations we are living in, nothing can be predicted easily. Anything can happen at any moment. We cannot predict things accurately. A small error can change the outcome. So unpredictability is a common principle in chaos.

Generators, attractors, and repellers:

The characteristics of the system, which have the potential to produce chaotic behavior, are called generators. In a long run, huge differences in the system can be caused by a very small difference in generators [68].

Sometimes, chaotic behavior inclines to some predictable behavior. In such situations, elements of the system bring chaotic aspects into a more simple and comprehensible pattern. These elements are attractors [66].

For a variety of initial conditions, a numerical set of values towards which the system tends to evolve is called as an attractor. Even if disturbed slightly, the system values remain close to the attractor. Otherwise, the set of numerical values is called a Repeller.

In biological studies, the behavior of animals is chaotic. Population experts try to understand this behavior over time which depends on several parameters such as resources available, the effect of infections, overcrowding, etc.

In situations of random behavior, chaos theory helps to draw sensible information.

As complex systems contain so much dynamics, computers are required to calculate various possibilities, and hence computational techniques will be widely useful.

2.9.2 Properties of Chaotic Systems:

The Chaotic theory is based on the property that complex behavior arises from the iterations of simple nonlinear rules on the system. The chaotic solution is always sensitive to initial conditions, bounded and aperiodic. The following are some interesting properties of chaotic systems:

(a) In phase space, due to exponential divergence and boundedness, stretching and folding of chaotic solutions always take place.

(b) These systems are either Dissipative or Conservative

(c) Chaotic systems can be controlled.

(d) Chaotic systems can be synced to one another. Multiple chaotic systems can be joined to bring a constant difference in their outputs. Outputs lag one another by a time lag and phase synchronization occurs. Their outputs are related to each other by an invertible map.

2.9.3 Routes to Chaos

The following are the major routes to Chaos

(a) The period-doubling route: Here, many sub-harmonic bifurcations occur within the system, which eventually accumulates at the critical value.

(b) The quasi-periodic route: This leads to chaos. In this, there occurs numerous periodic solutions with irrationally related frequencies.

(c) The intermittency route: Here the behavior is almost periodic. However, regimes of irregular behavior are observed.

2.10 Lyapunov function and Lyapunov stability

Lyapunov functions are very significant in the study of dynamical systems. While investigating the stability of ordinary differential equations, we construct certain scalar functions known as Lyapunov functions to prove equilibrium's stability [74]. The existence of Lyapunov functions is a necessary and sufficient condition for the stability of a certain class of ordinary differential equations.

Aleksandr Lyapunov in his thesis '*The general problem of stability of motion'* developed the global approach to the stability analysis of nonlinear dynamical systems[75]. He made comparisons with the popular method of linearization about the equilibrium points.

2.10.1 Definitions

For an autonomous nonlinear dynamical system

$$\dot{x} = f\big(x(t)\big), x(0) = x_0, \qquad \text{Eqn. (2.33)}$$

where $x(t) \in S \subseteq \mathbb{R}^n$denotes the system state vector, S and open set containing the origin, and $f: S \rightarrow \mathbb{R}^n$continuous on S. Let f has an equilibrium at x_e so that $f(x_e) = 0$ then

1. The equilibrium of Eqn. (3.33) is said to be Lyapunov stable, if $\forall\, \epsilon > 0, \exists\, \delta > 0$ such that, if $\|x(0) - x_e\| < \delta$, then for every $t \geq 0$, we have $\|x(t) - x_e\| < \epsilon$.

2. The equilibrium of Eqn. (3.33) is asymptotically stable of it is Lyapunov stable and $\exists\ \delta > 0 \ni$ if $\|x(0) - x_e\| < \delta$, then $\lim_{t\to\infty} \|x(t) - x_e\| = 0$.
3. The equilibrium of Eqn. (3.33) is exponentially stable if it is stable asymptotically and $\exists\ \{\alpha, \beta, \delta\} > 0 \ni$ if $\|x(0) - x_e\| < \delta$, then $\|x(t) - x_e\| \leq \alpha\|x(0) - x_e\|e^{-\beta t}, \forall\ t \geq 0$.

- At any equilibrium, if the solution close to the equilibrium remains close forever, then it is known as Lyapunov stability.
- If the solutions close to any equilibrium not only remain close but also, gradually converge to the equilibrium, it is called as asymptotically stable.
- If the solutions close to equilibrium points converge much rapidly towards the equilibrium, then it is called exponentially stable.

2.11 Basic Lyapunov theorem

Consider a non-negative locally positive definite function $V(x, t)$ whose derivative $\dot{V}$ is taken along the trajectories of the system. Then if,

- $\dot{V}(x, t) \leq 0$ locally in x and $\forall\ t$, then the system's origin is locally stable in the Lyapunov sense.
- $V(x, t)$ is also decrescent, and $\dot{V}(x, t) \leq 0$ locally in x and $\forall\ t$, then the system's origin is uniformly locally stable in the Lyapunov sense.
- $V(x, t)$ is also decrescent, and $-\dot{V}(x, t)$ is locally positive definite, then the system's origin is uniformly locally asymptotically stable.
- $V(x, t)$ is also decrescent, and $-\dot{V}(x, t)$ is positive definite, then the system's origin is globally uniformly asymptotically stable.

Along the system trajectories, the time derivative of V is given by the equation

$$\dot{V}\big|_{\dot{x}=f(x,t)} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f. \qquad \text{Eqn. (2.34)}$$

2.12 Motivation for Further Work

For the past few decades, dynamical systems theory has been attracting researchers from various disciplines whose aim is to understand the dynamic behavior of the systems and to gain some control over the systems on which they are working. The dynamical systems phenomena can be found in various disciplines like biology, physics, chemistry, engineering, finance, political science, and economics. Most of these systems are nonlinear and are studied by using ordinary differential equations. Discrete dynamical systems theory plays its role in understanding the dynamics of higher-order systems wherein the complicated behavior of higher-order systems can be studied by reducing the continuous problem into a discrete problem and thereby applying the techniques of discrete dynamical systems [59]. The study of dynamical systems introduced new theories like Chaos and Bifurcation.

The relevance of the dynamical systems theory comes from the conceptual and quantitative tools it offers. Most of the biological studies like chaos in cardiac rhythm, brain, or population dynamics can be brought into the framework of nonlinear dynamics for better treatment and understanding[81].

In the study of biological systems, the dynamics are often explored computationally through qualitative techniques. The solutions behavior at equilibria is investigated by linearization technique. The main goal is to observe whether the solutions of nonlinear systems at the origin resemble those of the linearized system. The study of equilibria and global nonlinear techniques plays a vital role in this study.

In literature, one can find many well-developed mathematical models in the domains of infectious diseases, predator/prey or host/parasite systems, competitive species, and competition & harvesting. Researchers from various domains study, implement and extend these models while investigating problems of their domain. Biological species always aim for their existence and engage in the quest for resources. Based on the application, prey-predator models assume various forms such as resource-consumer, host-parasite, tumor cells-immune systems, susceptibles-infectors, etc. As these processes involve loose-win-like situations, these models can be effectively implemented outside the ecosystems as well.

The wide variety of applications of dynamical system study of biological systems fascinated us and motivated us for the qualitative study of various mathematical models in biology.

2.13 Conclusion

Dynamical systems and theories like chaotic theory, the fractal theory has various applications in various domains of biology, medicine, and sciences. Scientists are not restricting their studies to simple, closed, and deterministic systems. They are aiming to understand the behavior of large complex open dynamical systems. Mathematicians are gaining insights into the systems from other domains, which in turn is helping them to develop fruitful results for understanding and explaining the nonlinear phenomena. All the methods discussed in this chapter help one to get a deep insight into a variety of dynamical behavior of real-world systems.

CHAPTER 3. DYNAMICS OF A THREE DIMENSIONAL CHAOTIC CANCER MODEL

3.1 Introduction

Cancer is a dangerous disease involving abnormal growth of cells with potential chances of diffusing to other body parts. In the present era, cancer is one of the major health concerns and is considered to be one of the main causes of death globally. Each year, millions of death cases, are reported due to various types of cancers. Statistics from international agencies such as Global Cancer Observatory (GCO) and World Health Organization (WHO) reveal that women generally fall victims to breast, colorectal, cervix, and stomach cancers while prostate, lung, and liver cancers are most common in men [82]–[84]. The life cycle of cells in multicellular organisms is regulated by apoptosis, a process for the elimination of cells, leading to cell death, through a sequence of programmed events. In this mechanism, there is no influence of harmful substances on the neighboring areas. Cancer cells skip this apoptosis mechanism, which allows them to grow beyond their natural capacity spreading into the other parts of the body. This process is termed metastasis, which brings death to cancer patients [85].

To understand the cancer dynamics, doctors usually rely on clinical trials. Clinical trials are helpful to determine whether new treatments are safe and effective and work better than the existing treatments in preventing cancer. Usually, clinical trials involve high costs and are time-consuming. So the specialists working on cancer treatments believe in developing alternative and feasible mechanisms for understanding the dynamics of cancer. There are very few ways of treating the disease like surgery and radiotherapies. In many cases, the chances of survival are very less. For the survival of patients, the amount of administrated therapy is very significant. During the therapy, there is a danger to healthy cells. Sometimes, they are killed along with unwanted tumor cells in which the situation becomes more serious for the patient. The therapy does not only kill the tumor cells but also kills some healthy tissues resulting in serious damage. Many of the curative techniques are still in their infancy. The need to address the preventive measures and successful treatment strategies are necessarily required. Mathematical modeling helps us to understand the short and long-term dynamics of cancer. Numerous models have been developed to understand the dynamic interactions of tumor growth, immune system, and different curing methods such as immunotherapy or chemotherapy.

There are investigations using cellular automata for the tumor growth models. These comprise very precise patient characteristics, tumors, and drugs which are presented in the models [60], [86], [87]. Both PDE and cellular automata approach for modeling growth of a tumor was used by Anderson and Chaplin; Enderling et al [88], [89]. de Pillis and Radunskaya in their work constructed a generalized growth model of a tumor using ordinary differential equations [10]. In this model, they studied the dynamics of tumor by means of healthy, tumor, and immune cells. Prey-predator models were successfully implemented by several researchers for studying the interactions between tumor and immune cells [8]–[15].

Inside the human body, the interactions in the immune system with the target populations of bacteria, viruses, tumor cells are considered dynamical processes. It is argued that tumor growth is extremely sensitive to the initial conditions and hence is classified as a chaotic dynamical system. Various chaotic models have been proposed to well fit the observations. In the present study, a three-dimensional model of cancer tumor growth involving the interactions between tumor, healthy tissue, and activated immune system cells is considered. The dynamics of the model are explored by performing computationally the local equilibria stability and established the existence of chaos in the proposed model.

3.2 The mathematical model

The model equations developed are

$$\begin{cases} \frac{dx}{dt} = \alpha x(1-y)(1+z) - x^2 y \\ \frac{dy}{dt} = \beta y(1-z)(1+x) - y^2 z \\ \frac{dz}{dt} = \gamma z(1-x)(1+y) - z^2 x \end{cases} \qquad \text{Eqn. (3.1)}$$

where $x(t), y(t)$ and $z(t)$ are the number of tumor cells, healthy host cells, and effector immune cells at time t in a single tumor – site compartment.

The parameters α, β and γ belonging to $\mathbb{R}$ are non-negative and arbitrary representing the per capita growth rates of tumor, healthy host, and effector immune cells respectively.

It is an idealized model demonstrating how varieties of immune responses are generated by the combination of nonlinear interactions proposed among the immune system and its targets. The solutions of the model correspond to three different states namely virgin state, immune state, and state of tolerance in the terminology of an immunologist. It replicates the primary and secondary responses. In addition, for the transplanted tumor cells, the model predicts the threshold level. The tumor graft is rejected or the host eliminates the infectious species below this threshold. Sometimes, the tumor cells present in the bloodstream are very significant for the study. Such situations arise in cases like leukemia. ODEs can be successfully implemented for such studies. The present model can be considered for developing new administration methodologies for drugs in cancer treatment. The details like the effects of drugs administrated on the metabolism of patients are very helpful.

3.3 Local Asymptotic Stability of Equilibria

The equilibria are computed by solving $\dot{X} = F(X) = 0$, where $X = (x, y, z)^T$ and $F = [f, g, h]^T$ and

$f(x, y, z) = \alpha x(1 - y)(1 + z) - x^2 y$, $g(x, y, z) = \beta y(1 - z)(1 + x) - y^2 z$ and $h(x, y, z) = \gamma z(1 - x)(1 + y) - z^2 x$.

The following four equilibria are obtained analytically

- $(x, y, z) = (0,0,0)$
- $(x, y, z) = \left(0, -1, \frac{\beta}{\beta - 1}\right)$ $if\ \beta \neq 1$
- $(x, y, z) = \left(\frac{\gamma}{\gamma - 1}, 0, -1\right)$ if $\gamma \neq 1$
- $(x, y, z) = \left(-1, \frac{\alpha}{\alpha - 1}, 0\right)$ $if\ \alpha \neq 1$

One last equilibrium corresponding to the case $\{x, y, z\} \neq 0$ must be considered, but it has no analytical expression. It could be found from the numerical solution of the equations below.

$$\begin{cases} 1 - y + z - yz - \dfrac{xy}{\alpha} = 0 \\ 1 - z + x - zx - \dfrac{yz}{\beta} = 0 \\ 1 - x + y - xy - \dfrac{zx}{\gamma} = 0 \end{cases}$$

For studying the equilibrium point stability, we must choose an equilibrium point and find the Jacobian at that equilibrium point and check its eigenvalues. For each eigenvalue, if the real part is negative, then there is a stable equilibrium. We get an unstable equilibrium if there is at least one eigenvalue whose real part is positive. If one of the eigenvalues is 0, then that equilibrium point may be stable or unstable.

3.3.1 Stability at $(\mathbf{0},\mathbf{0},\mathbf{0})$

The Jacobian $\frac{\partial F}{\partial x}$ about the equilibrium (0,0,0) is

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}$$

Theorem 3.1: The equilibrium (0,0,0) is locally asymptotically stable if all the parameters α, β, and γ are negative.

Proof. The eigenvalues of the Jacobian about (0,0,0) are α, β, and γ. All the parameters are real and if they are negative then the equilibrium (0,0,0) is locally asymptotically stable.

A pair of 50 initial values are taken where all the three parameters are taken as $\alpha = -0.1334, \beta = 0.1024$, and $\gamma = -0.9591$. The corresponding trajectory plots are given in the following Fig. 3.1.1 and Fig. 3.1.2

Since α, β and γ are meant to be positive and hence the equilibrium (0,0,0) is practically not feasible.

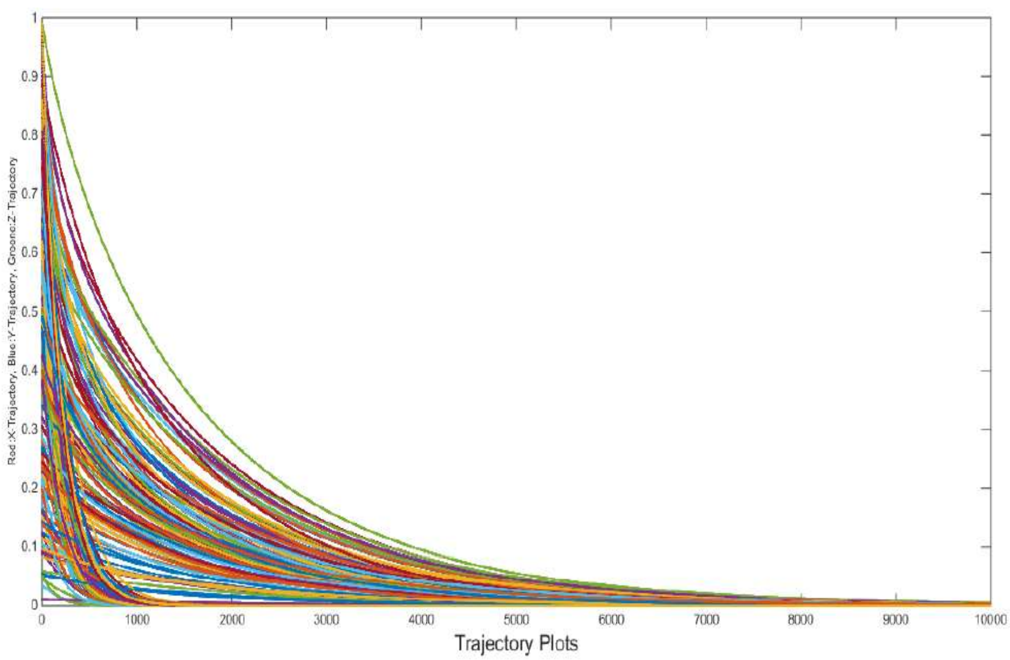


Fig. 3. 1.1 Trajectory plots which are attracting towards the origin for 50 different initial values with negative parameters.

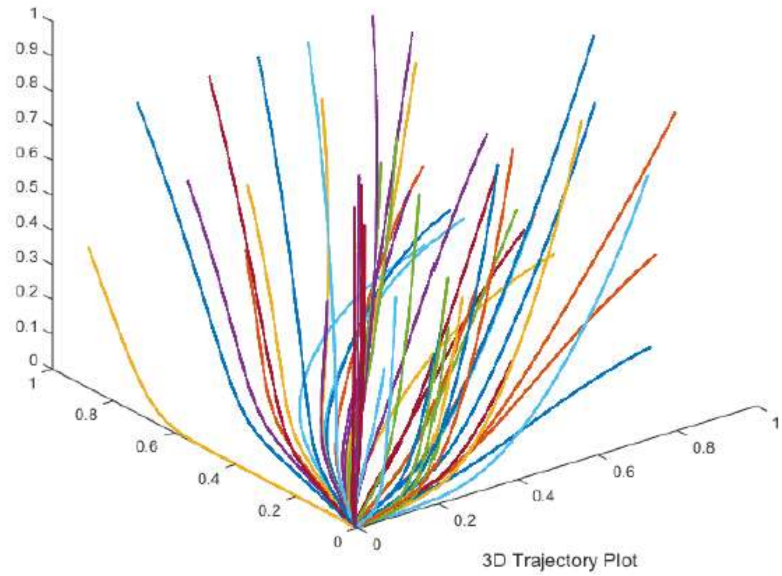


Fig. 3.1.2. The corresponding 3D version of the trajectory plots of Fig 3.1.1.

Around the neighborhood of the equilibrium point (0,0,0), a pair of 20 dissimilar initial conditions were taken. Here the three parameters are taken as $\alpha = 0.7455, \beta = 0.7363$, and $\gamma = 0.5619$ whose trajectories are plotted in the following Fig. 3.2.1 and Fig. 3.2.2.

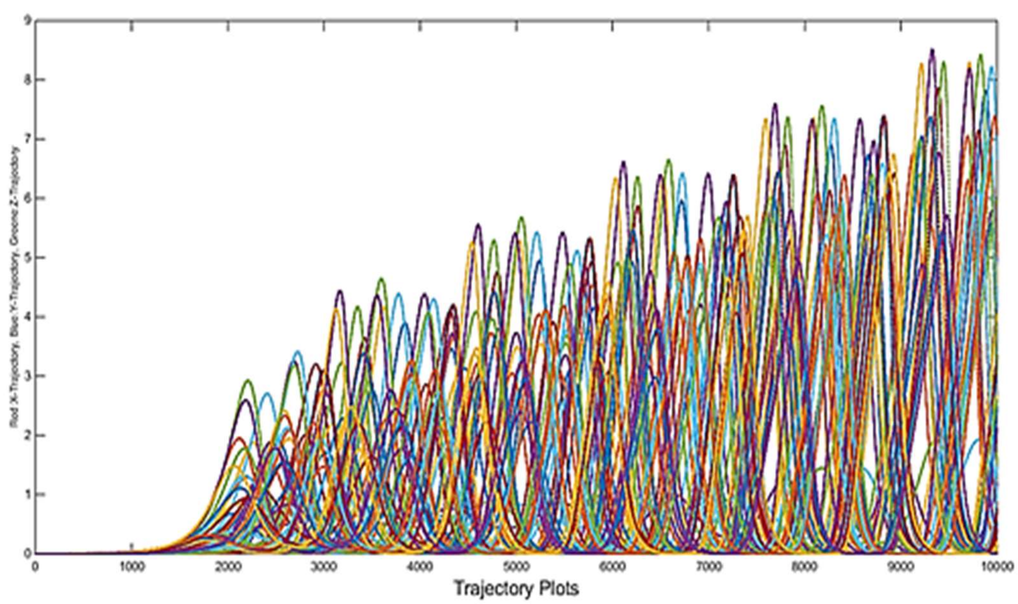


Fig.3. 2.1 Trajectory plots which are repelling towards the origin for 20 different initial values with positive parameters $\boldsymbol{\alpha = 0.7455}$, $\boldsymbol{\beta = 0.7363}$, and $\boldsymbol{\gamma = 0.5619}$.

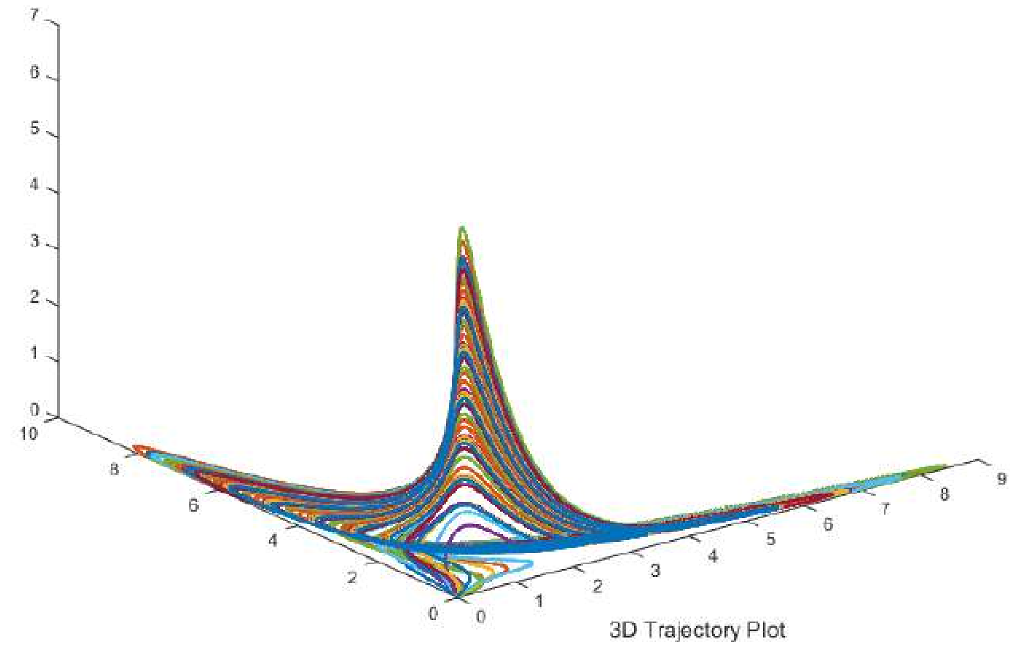


Fig.3.2.2. The corresponding 3D version of the trajectory plot of Fig 3.2.1.

3.3.2 Local Stability of $(\mathbf{0}, -\mathbf{1}, \boldsymbol{\beta}/(\boldsymbol{\beta}-\mathbf{1}))$

The Jacobian $\frac{\partial F}{\partial x}$ about the equilibrium $(0, -1, \frac{\beta}{\beta-1})$ is

$$\begin{pmatrix} 2\alpha\left(\frac{\beta}{\beta-1}+1\right) & 0 & 0 \\ \beta\left(\frac{\beta}{\beta-1}-1\right) & \frac{2\beta}{\beta-1}-\beta\left(\frac{\beta}{\beta-1}-1\right) & \beta-1 \\ -\frac{\beta^2}{(\beta-1)^2} & \frac{\beta\gamma}{\beta-1} & 0 \end{pmatrix}$$

Theorem 3.2: The equilibrium $(0, -1, \frac{\beta}{\beta-1})$ is locally asymptotically stable if

$$\alpha > \frac{1}{2}, \frac{2\alpha-1}{4\alpha-1} < \beta < \frac{1}{2}, 0 < \gamma < \frac{2\beta-1}{\beta^2-\beta}$$

A comprehensive list of parameters α, β and γ satisfying the condition for local asymptotic stability of $(0, -1, \frac{\beta}{\beta-1})$ are fetched. The parameters are plotted in Fig. 3.3.

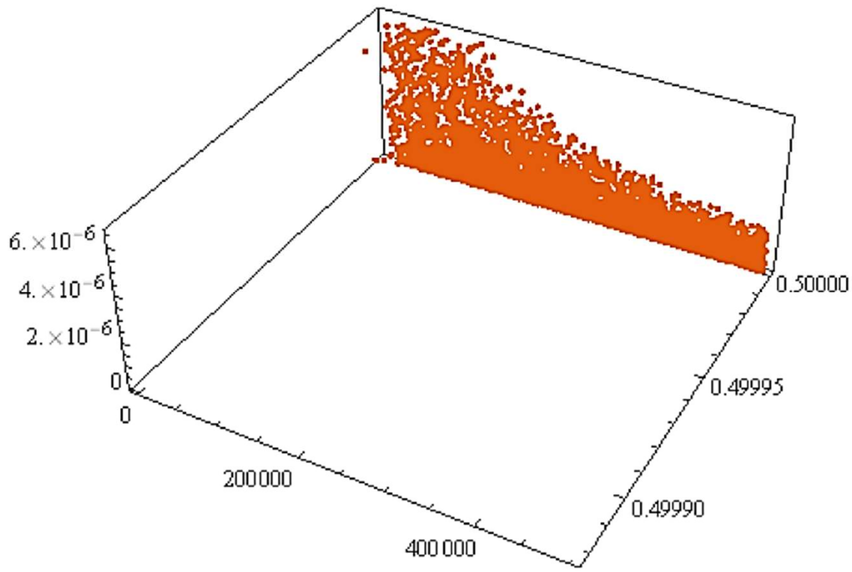


Fig. 3. 3. Parameter space $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ for which the fixed point $(\mathbf{0}, -\mathbf{1}, \frac{\boldsymbol{\beta}}{\boldsymbol{\beta}-\mathbf{1}})$ is attracting.

A pair of 50 different initial values are taken where the three parameters are taken as $\alpha = -0.1334, \beta = -0.1024$, and $\gamma = -0.9591$ whose trajectories are plotted in the following Fig. 3.4.1 and 3.4.2.

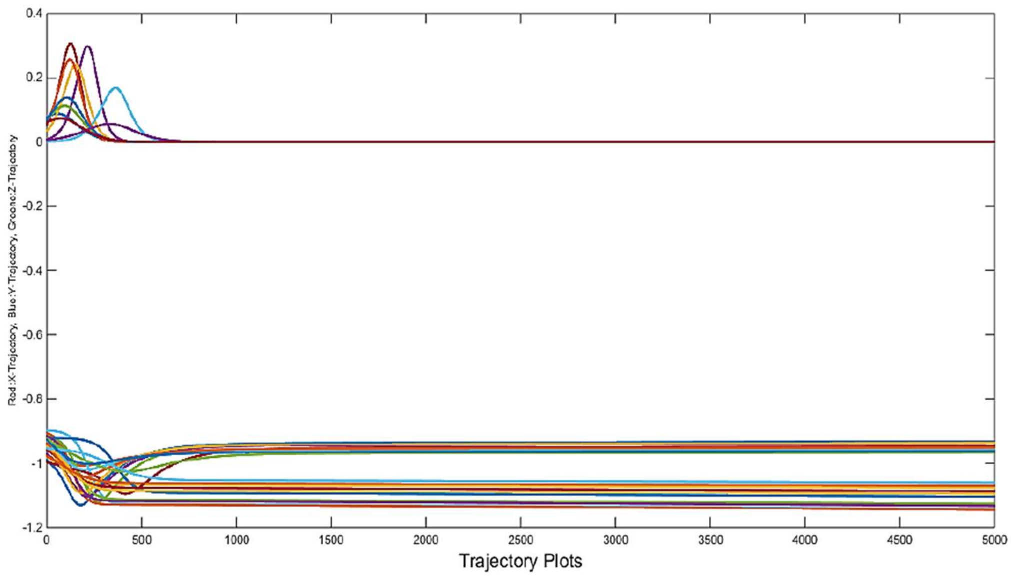


Fig.3.4.1 Trajectory plots which are attracting towards the fixed point $\left(0,-1,\frac{\beta}{\beta-1}\right)$ for 50 different initial values.

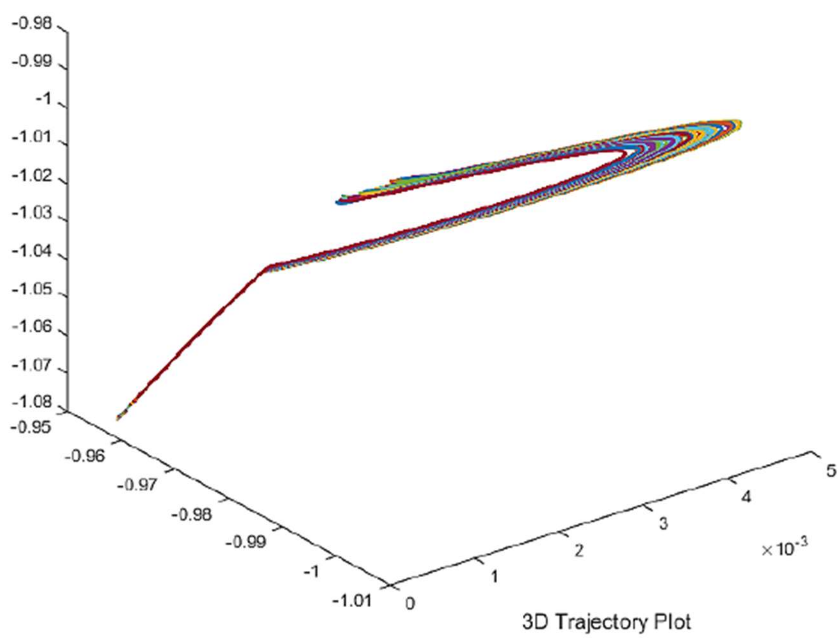


Fig.3.4.2 The corresponding 3D trajectory plot of Fig. 3.4.1.

A pair of 50 initial values are taken where all the three parameters are taken as $\alpha = 0.9064, \beta = 0.3927,$ and $\gamma = 0.0249$. The corresponding repelling trajectories are depicted in the following Fig. 3.5.1 and Fig 3.5.2.

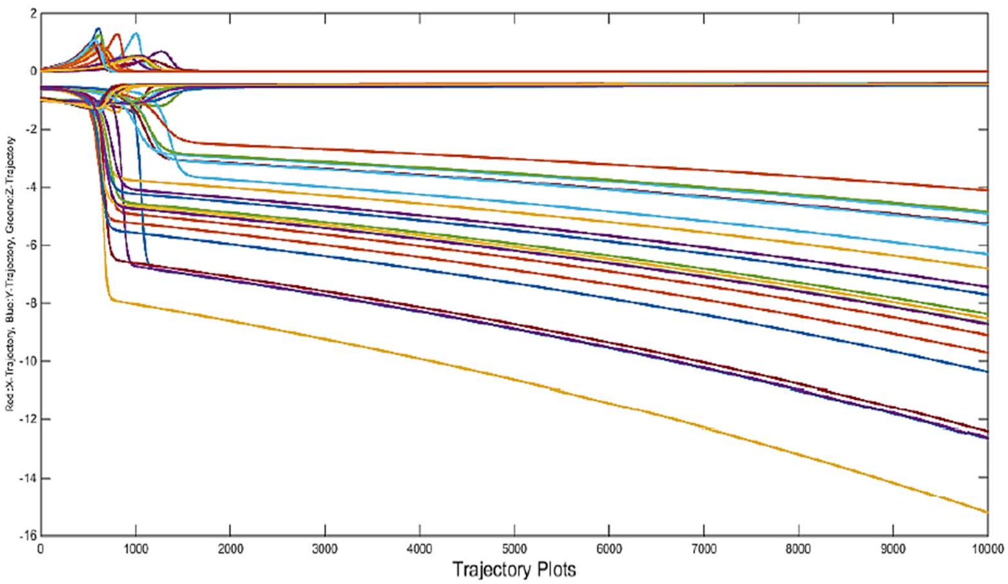


Fig. 3. 5.1. Trajectory plots which are repelling from $(\mathbf{0}, -\mathbf{1}, \frac{\boldsymbol{\beta}}{\boldsymbol{\beta}-\mathbf{1}})$ for 50 initial values.

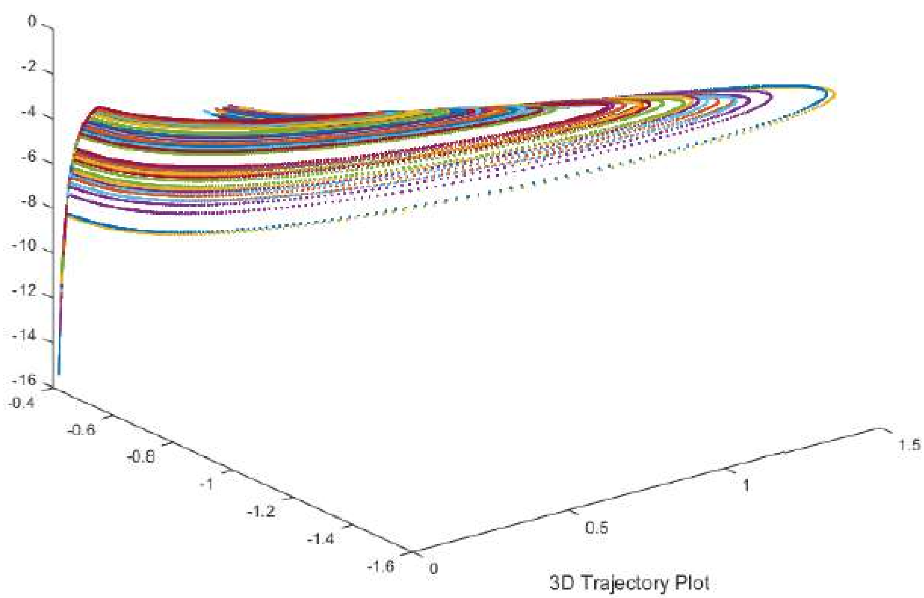


Fig. 3.5.2. The corresponding 3D trajectory plot of Fig. 3.5.1.

3.3.3 Local Stability of $(\gamma/(\gamma-1), 0, -1)$

The Jacobian $\frac{\partial F}{\partial x}$ about the equilibrium $(\frac{\gamma}{\gamma-1}, 0, -1)$ is

$$\begin{pmatrix} 0 & -\frac{\gamma^2}{(\gamma-1)^2} & \frac{\alpha\gamma}{\alpha-1} \\ 0 & 2\beta\left(\frac{\gamma}{\gamma-1}+1\right) & 0 \\ \gamma-1 & \gamma\left(\frac{\gamma}{\gamma-1}-1\right) & \frac{2\gamma}{\gamma-1}-\gamma\left(\frac{\gamma}{\gamma-1}-1\right) \end{pmatrix}$$

Theorem 3.3: The locally asymptotic stability condition for the equilibrium $(\frac{\gamma}{\gamma-1}, 0, -1)$ is

$$0 < \beta \leq \frac{1}{2}, 0 < \gamma < \frac{\alpha+2}{2\alpha} - \frac{1}{2}\sqrt{\frac{\alpha^2+4}{\alpha^2}}$$

A comprehensive list of parameters α, β and γ are fetched satisfying the local asymptotic stability condition of $(\frac{\gamma}{\gamma-1}, 0, -1)$. The parameters are plotted in Fig. 3.6.

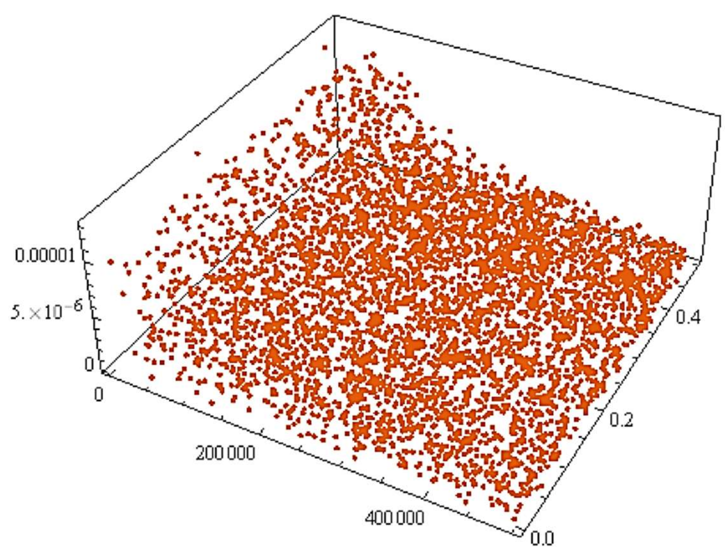


Fig. 3. 6. Parameter space (α, β, γ) for which the fixed point $(\frac{\gamma}{\gamma-1}, 0, -1)$ is attracting.

A pair of 50 different initial values are taken where the three parameters are taken as $\alpha = -0.1334, \beta = -0.1024$, and $\gamma = -0.9591$ whose trajectories are depicted in Fig. 3.7.1, and Fig. 3.7.2.

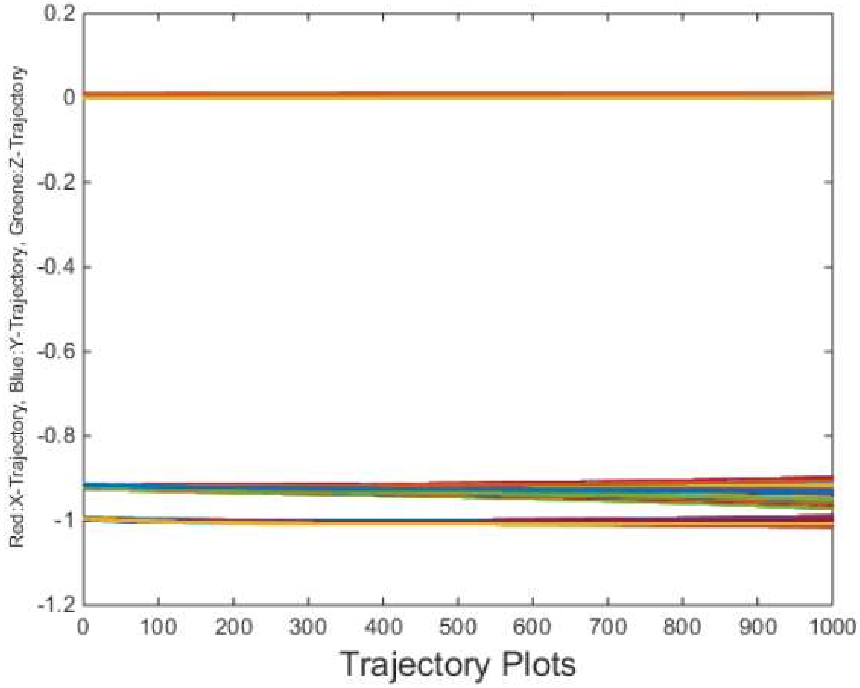


Fig.3.7.1. Trajectory plots which are attracting towards the fixed point $(\frac{\gamma}{\gamma-1}, 0, -1)$ for 50 different initial values.

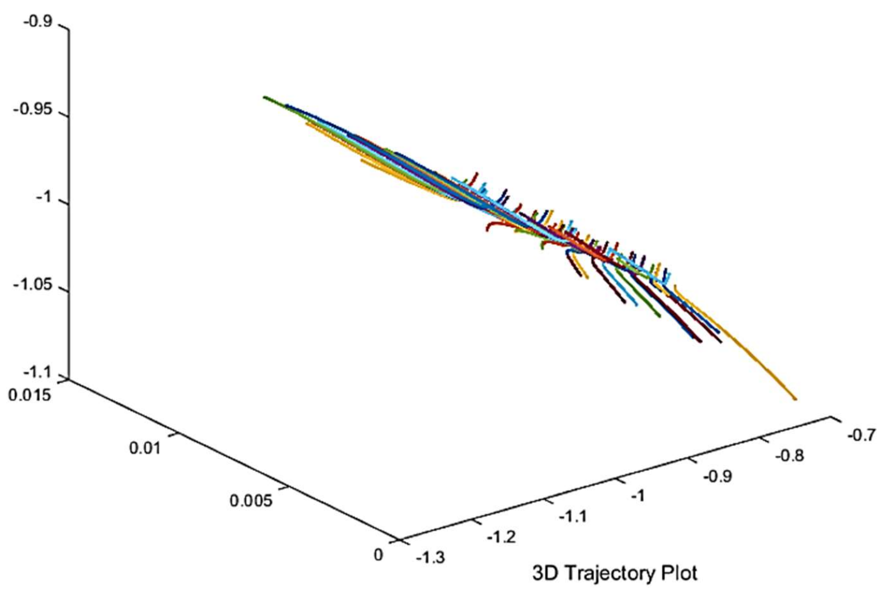


Fig.3. 7. 2. The corresponding 3D trajectory plot of Fig. 3.7.1.

The fixed point $(0,-1,\frac{\beta}{\beta-1})$ does repel. When $(\alpha = 0.0350, \beta = 0.4402, \gamma = 0.4398)$ and $(\alpha = 0.775, \beta = 0.6184, \gamma = 0.9385)$ for all initial values taken in the neighborhood of $(0,-1,\frac{\beta}{\beta-1})$. The trajectories are repelling as shown in Fig. 3.8.1, Fig 3.8.2, Fig 3.8.3, and Fig 3.8.4 respectively.

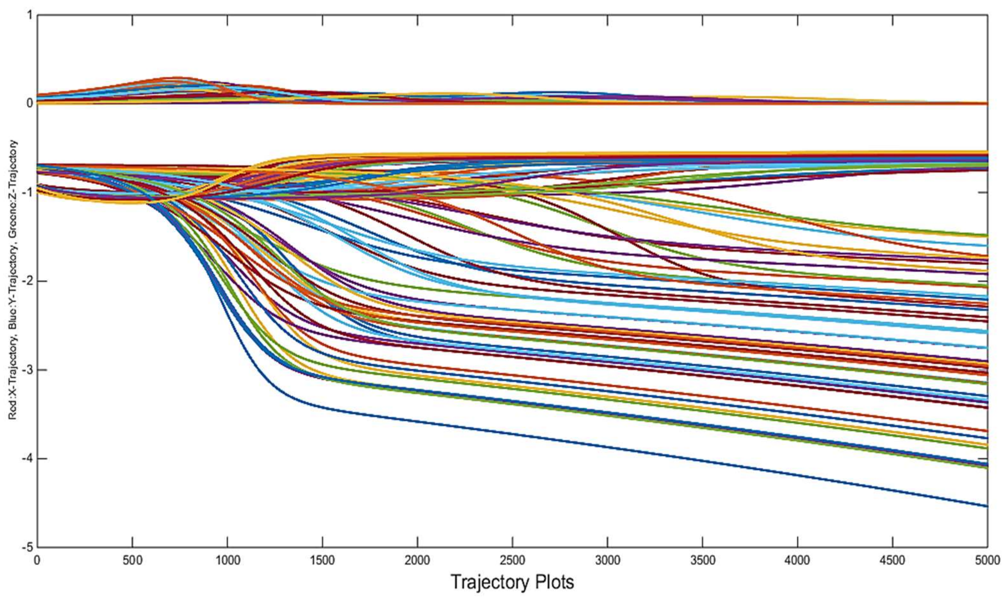


Fig.3.8.1 Trajectory plots which are repelling from the fixed point $(\mathbf{0},-\mathbf{1},\frac{\boldsymbol{\beta}}{\boldsymbol{\beta}-\mathbf{1}})$ for 50 different initial values for $\boldsymbol{\alpha} = \mathbf{0.0350}, \boldsymbol{\beta} = \mathbf{0.4402}, \boldsymbol{\gamma} = \mathbf{0.4398}$.

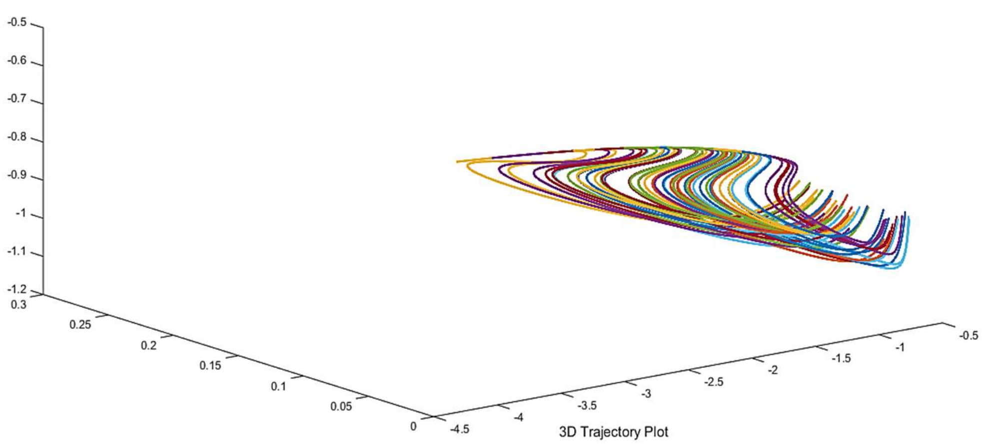


Fig.3.8.2. The corresponding 3D trajectory plot of Fig 3.8.1.

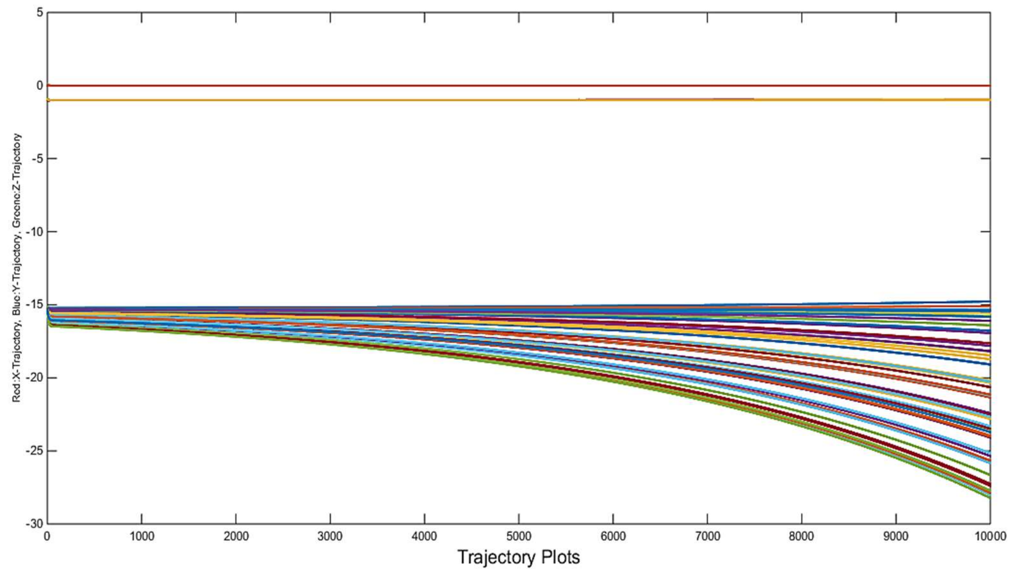


Fig.3.8.3 Trajectory plots which are repelling from the fixed point $(0, -1, \frac{\beta}{\beta-1})$ for 50 different initial values for $\alpha = 0.775, \beta = 0.6184, \gamma = 0.9385$.

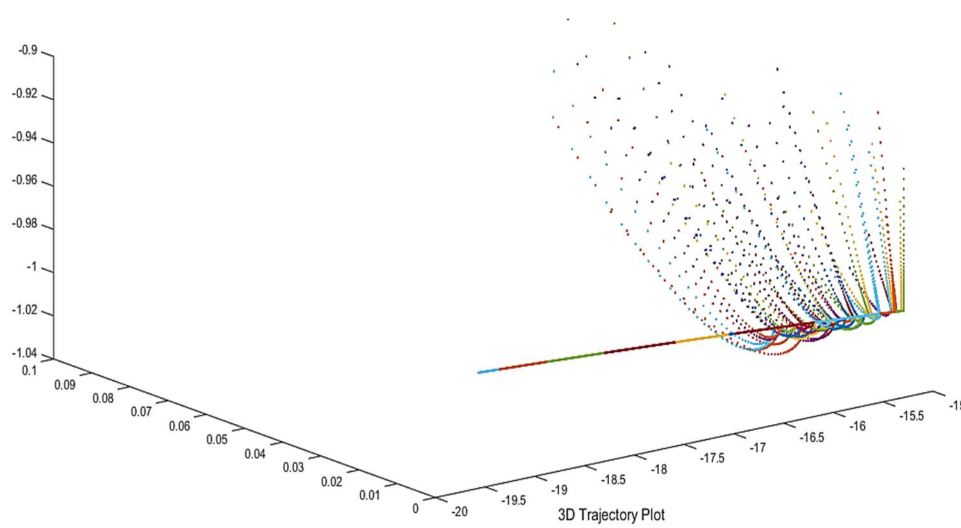


Fig.3.8.4. The corresponding 3D trajectory plot of Fig 3.8.3.

3.3.4 Local Stability of $(-\mathbf{1}, \boldsymbol{\alpha}/(\boldsymbol{\alpha}-\mathbf{1}), \mathbf{0})$

The Jacobian $\frac{\partial F}{\partial x}$ about the equilibrium $(-1, \frac{\alpha}{\alpha-1}, 0)$ is

$$\begin{pmatrix} 0 & 0 & 0 \\ \frac{2\alpha\beta}{\alpha-1} & \frac{2\alpha}{\alpha-1}+2\beta & -\frac{\alpha\left(\frac{\alpha}{\alpha-1}+\beta\right)}{\alpha-1} \\ \frac{\alpha\gamma}{\alpha-1}+\gamma-1 & -\gamma & \left(\frac{\alpha}{\alpha-1}+1\right)\gamma \end{pmatrix}$$

Theorem 3.4: The local asymptotic stability condition for the equilibrium $(-1, \frac{\alpha}{\alpha-1}, 0)$ is given by

$$\alpha = \frac{1}{2}, \frac{1}{2} < \beta < 1, 0 < \gamma < \frac{1-2\beta}{\beta-1}$$

A comprehensive list of parameters α, β and γ are fetched satisfying the local asymptotic stability condition of $(-1, \frac{\alpha}{\alpha-1}, 0)$. The parameters are plotted in Fig. 3.9.

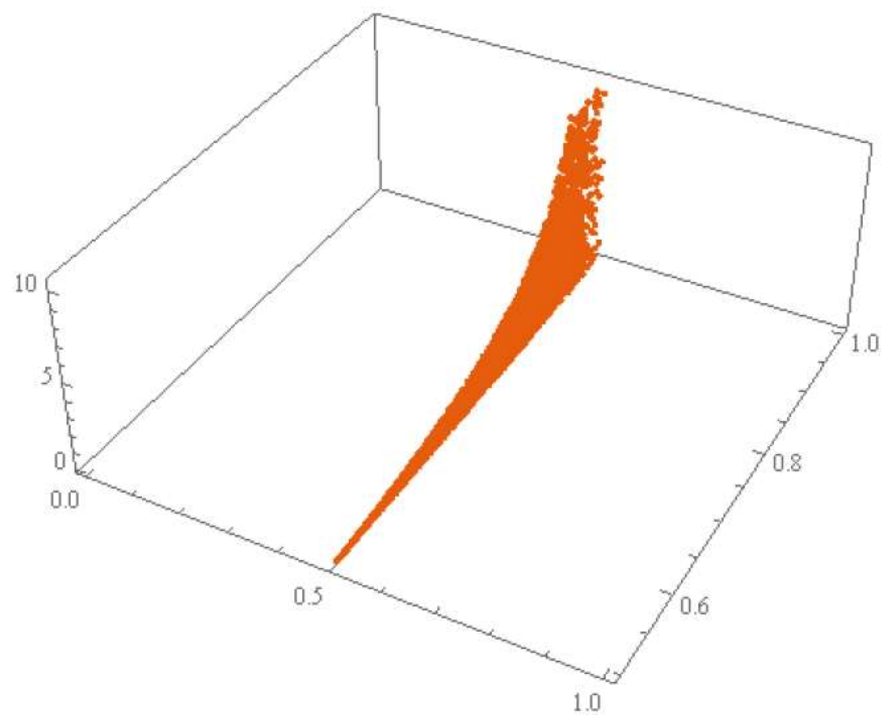


Fig. 3.9. Parameter space $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ for which the fixed point $(-\mathbf{1}, \frac{\alpha}{\alpha-1}, \mathbf{0})$ is attracting.

A pair of 50 different initial values are taken where the three parameters are taken as $\alpha = 0.5, \beta = 0.5,$ and $\gamma = 0.00039$ whose trajectories are plotted in Fig. 3.10.1, and Fig. 3.10.2

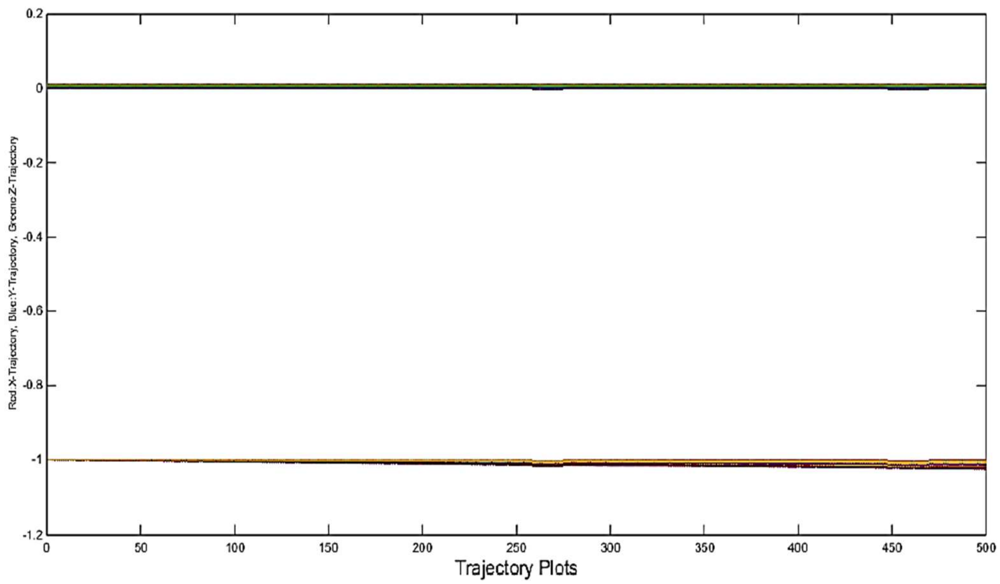


Fig.3.10.1 Trajectory plots which are attracting towards the fixed point $(-1, \frac{\alpha}{\alpha-1}, 0)$ for 50 different initial values.

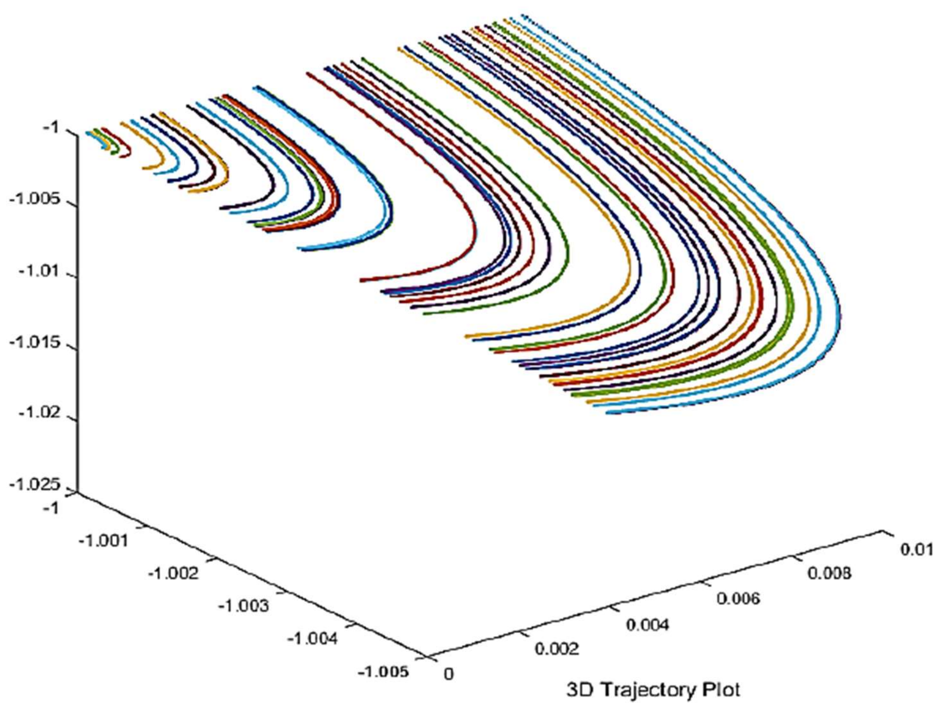


Fig.3.10.2. The corresponding 3D trajectory plot of Fig 3.10.1.

When $(\alpha = 0.8149, \beta = 0.3496, \gamma = 0.0216)$ and $(\alpha = 0.4049, \beta = 0.4824, \gamma = 0.0639)$ for all initial values taken in the neighborhood of $(-1, \frac{\alpha}{\alpha-1}, 0)$, the trajectories are repelling as shown in figures Fig. 3.11.1-Fig. 3.11.4 respectively.

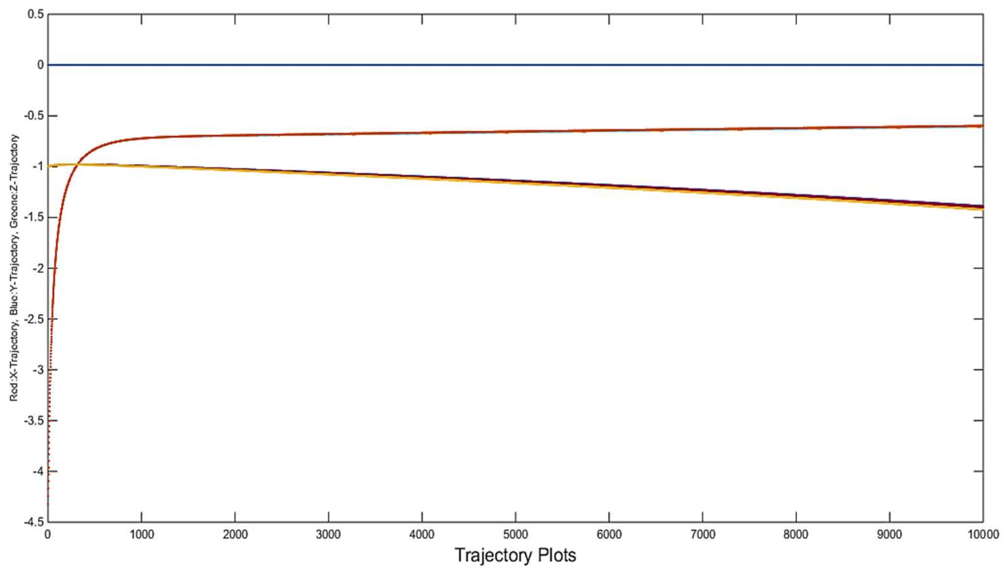


Fig.3.11.1 Trajectory plots which are repelling from the fixed point $(-1, \frac{\alpha}{\alpha-1}, 0)$ for 50 different initial values for $\boldsymbol{\alpha = 0.8149, \beta = 0.3496, \gamma = 0.0216}$.

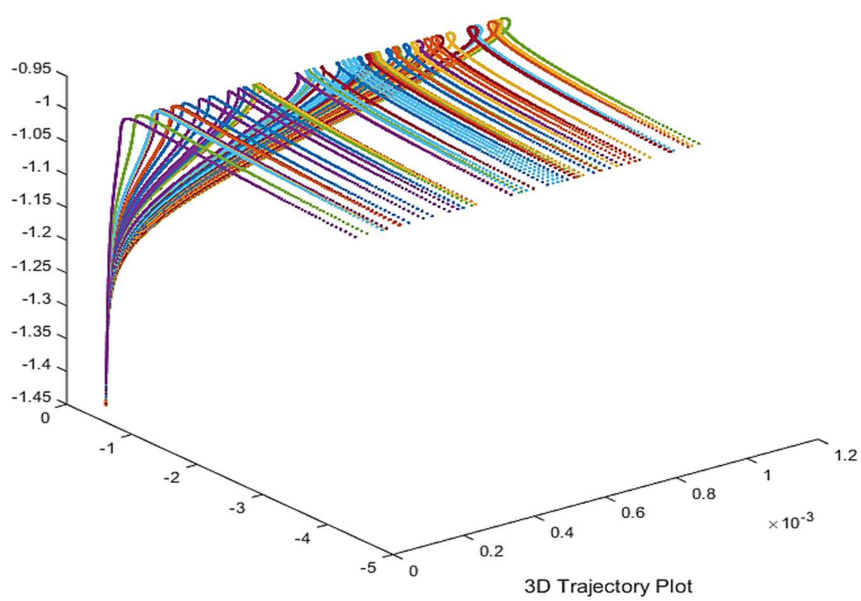


Fig.3.11.2. The corresponding 3D trajectory plot of Fig 3.11.1.

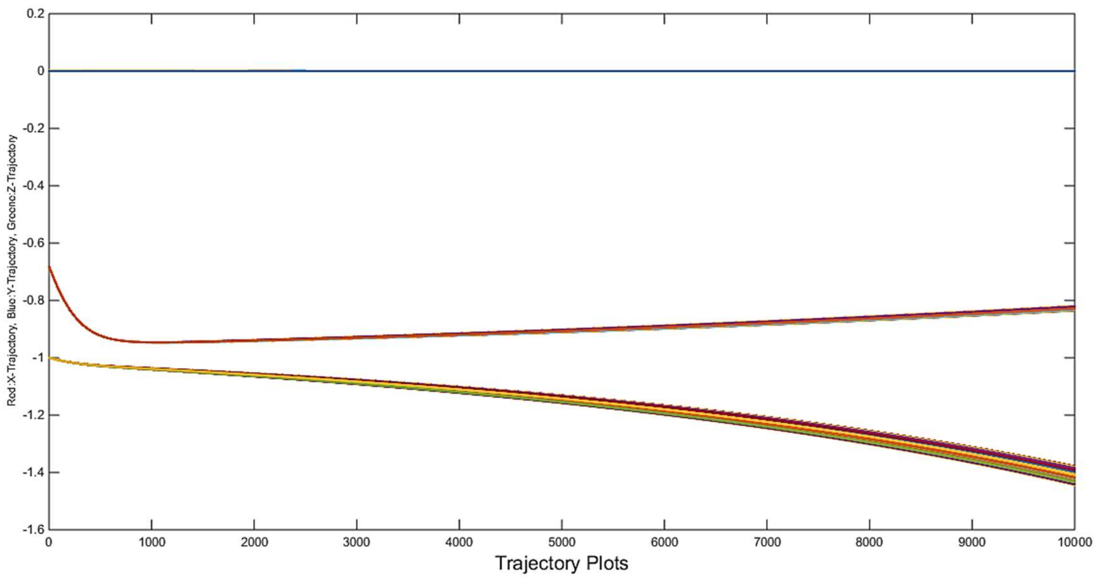


Fig.3.11.3 Trajectory plots which are repelling from the fixed point $(-1,\frac{\alpha}{\alpha-1},0)$ for 50 different initial values for $\boldsymbol{\alpha=0.4049,\beta=0.4824,\gamma=0.0639}$.

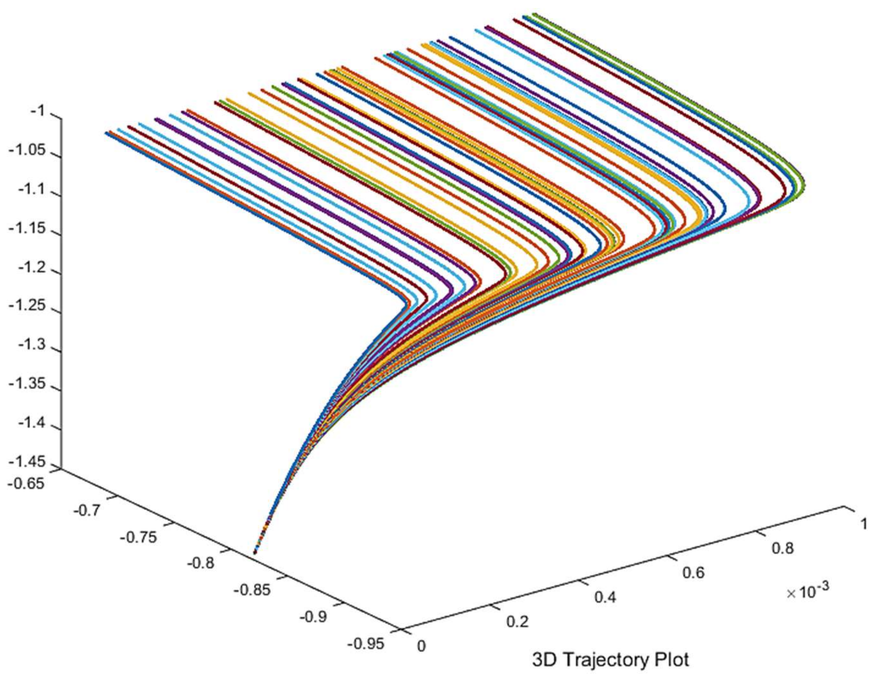


Fig.3.11.4. The corresponding 3D trajectory plot of Fig 3.11.3.

3.4 Chaotic solutions

The study reveals that the system exhibits chaos with the selected parameters. It has been observed that whenever the repelling of equilibrium (0,0,0) happens, then chaos occurs in the system for any initial conditions considered in the neighborhood of (0,0,0). From this observation, the following theorem was considered.

Theorem 3.5. The system exhibits a chaotic solution for all positive values of the parameters α, β, and γ.

A region of parameters $(\alpha, \beta, \gamma) \subset \mathbb{R}^3$ is estimated, where the system possesses chaotic solutions. The region is shown in Fig. 3.12. The parameters and the corresponding three-dimensional chaotic trajectories for 50 different initial values from the neighborhood of the origin are shown in the figures Fig. 3.13.1 – Fig. 3.13.3. Here different colors in the chaotic attractor correspond to 50 different sets of initial values.

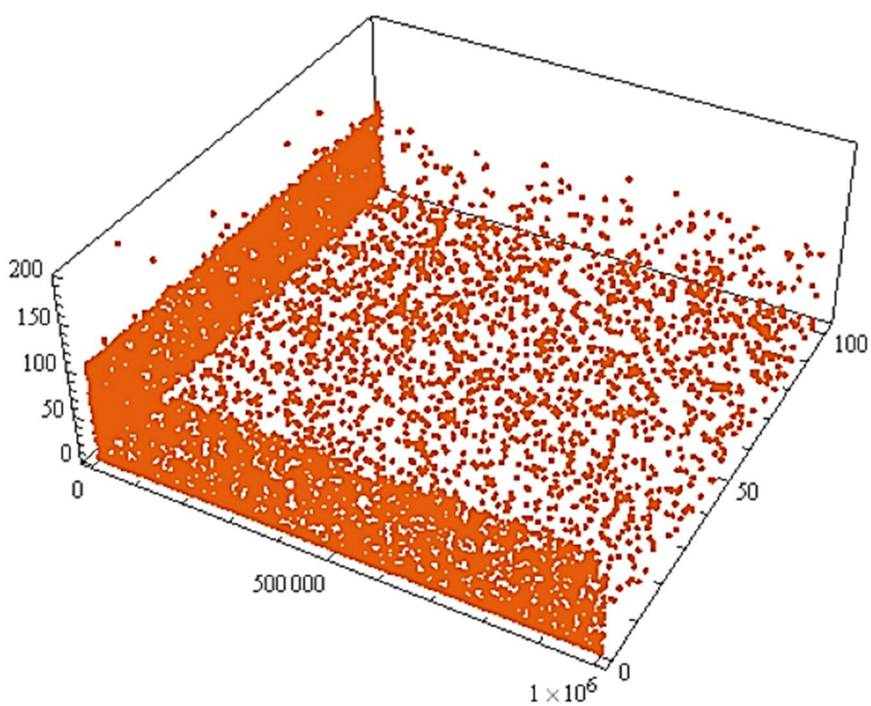


Fig. 3.12. Region of parameters $(\alpha, \beta, \gamma) \subset \mathbb{R}^3$ for which the solution is chaotic.

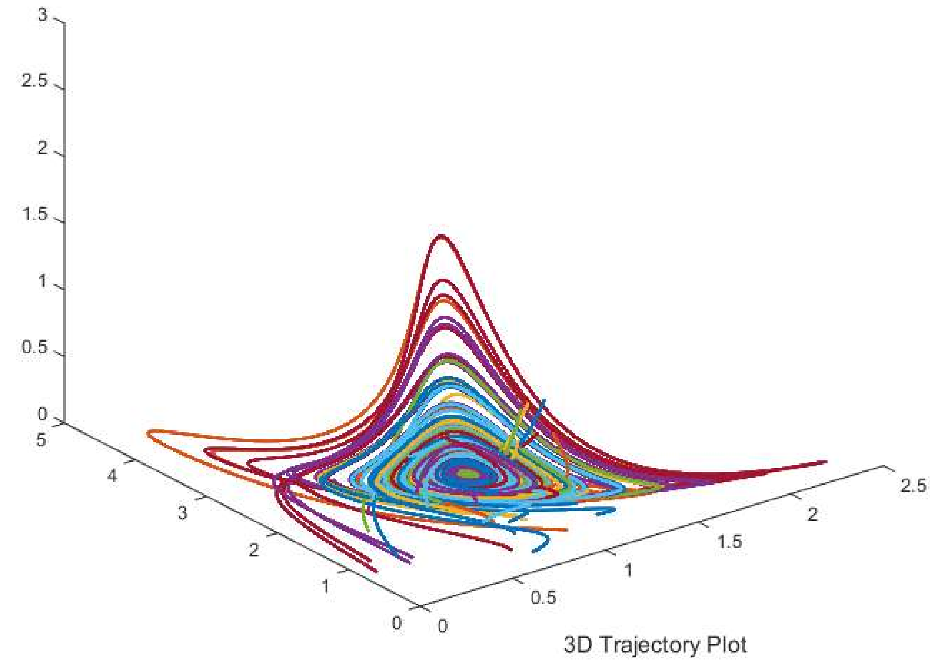


Fig. 3.13.1 Chaotic trajectory plots which are repelling from the fixed point $(0,0,0)$ for $\alpha = 0.2834, \beta = 0.6825$ and $\gamma = 0.3581$.

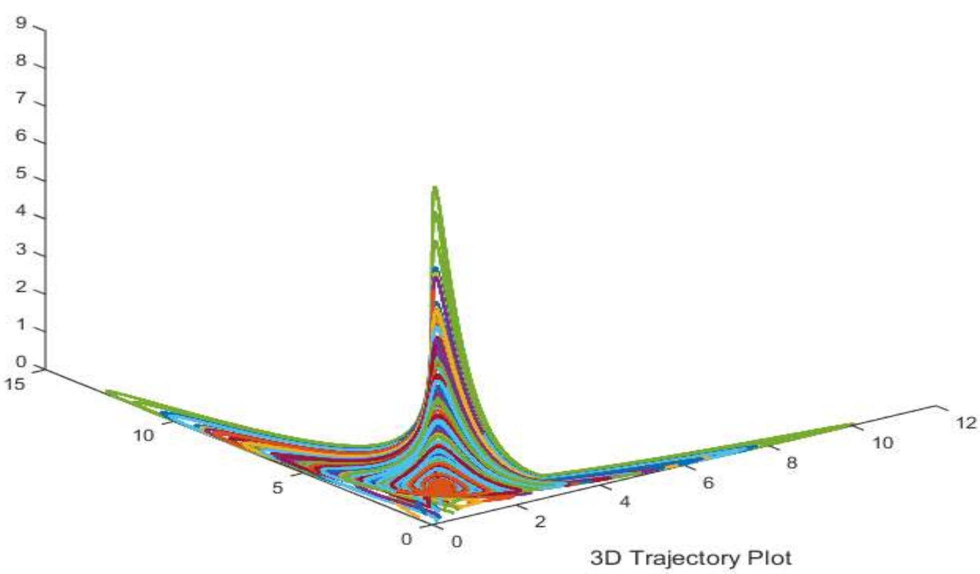


Fig. 3.13.2 Chaotic trajectory plots which are repelling from the fixed point $(0,0,0)$ for $\alpha = 0.6658, \beta = 0.9733$ and $\gamma = 0.6277$.

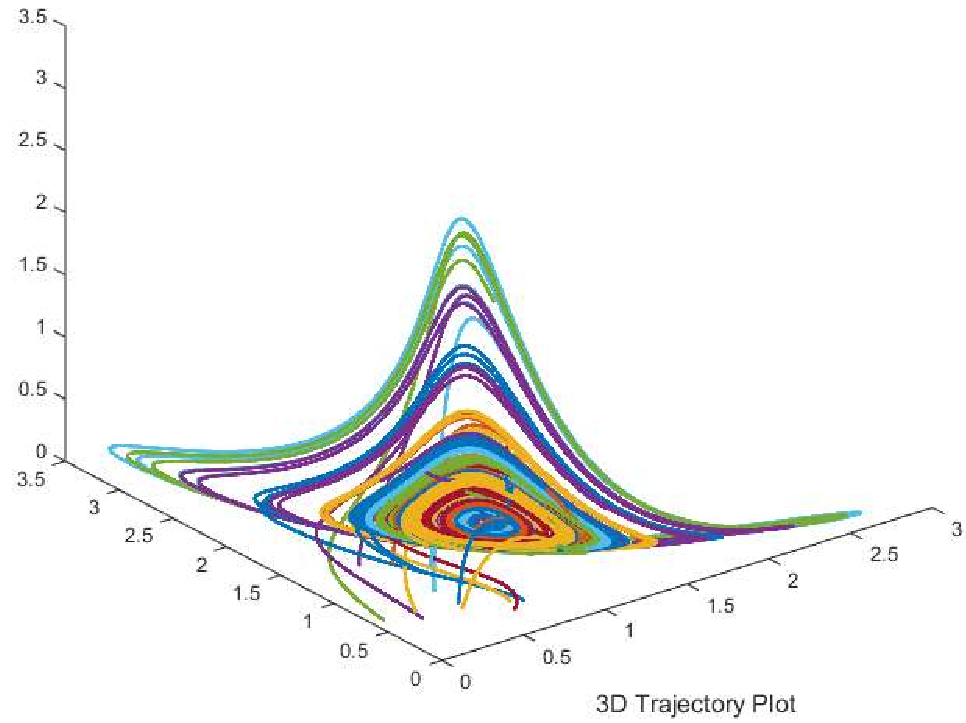


Fig. 3.13.3 Chaotic trajectory plots which are repelling from the fixed point $(\mathbf{0}, \mathbf{0}, \mathbf{0})$ for $\boldsymbol{\alpha} = \mathbf{0.3440}, \boldsymbol{\beta} = \mathbf{0.5337}$ and $\boldsymbol{\gamma} = \mathbf{0.6278}$.

The two-dimensional projections in the xy, yz and zx planes are plotted in Fig 3.13.4 – Fig. 3.13.6, for all the three chaotic attractors as stabled in Fig. 3.13.1 – Fig. 3.13.3. The trajectory plot up to 10^5 iterations are given for 50 different initial values for each of the set of parameters as given in Fig. 3.13.1 – Fig. 3.13.3

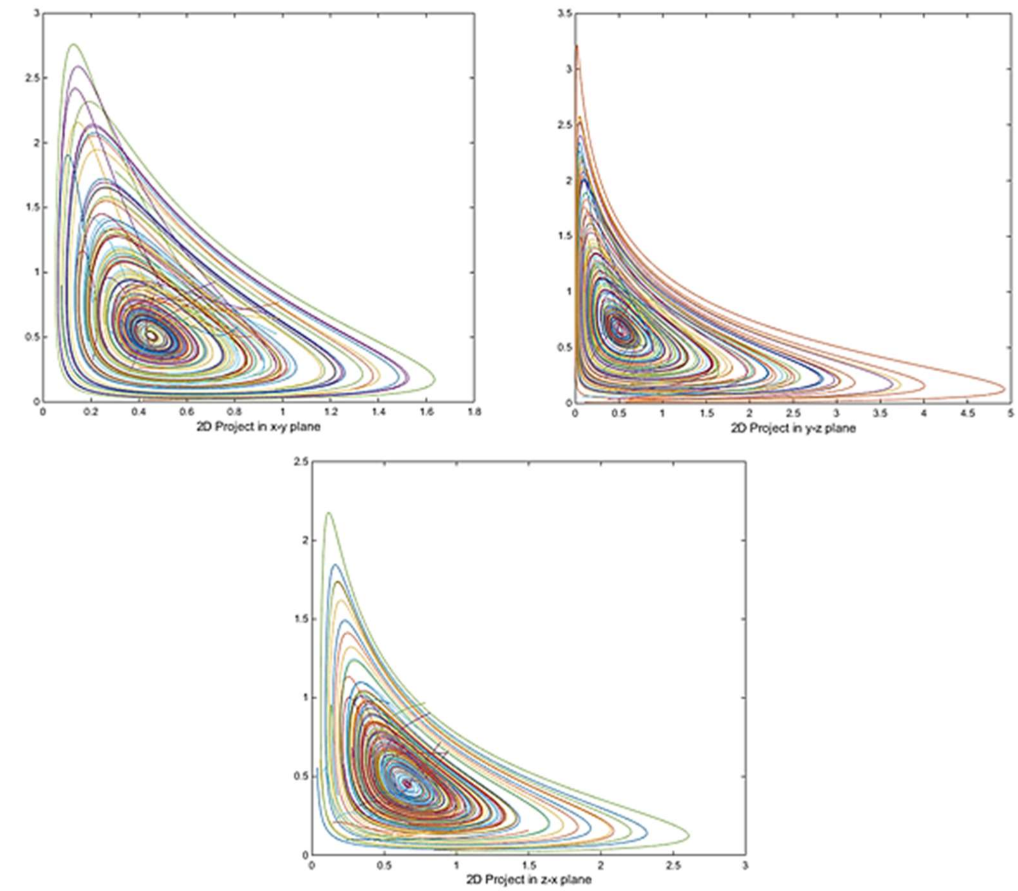


Fig.3.13.4 Projection of chaotic attractor in xy, yz and zx planes for parameter values $\alpha = 0.2834, \beta = 0.6825$ and $\gamma = 0.3581$.

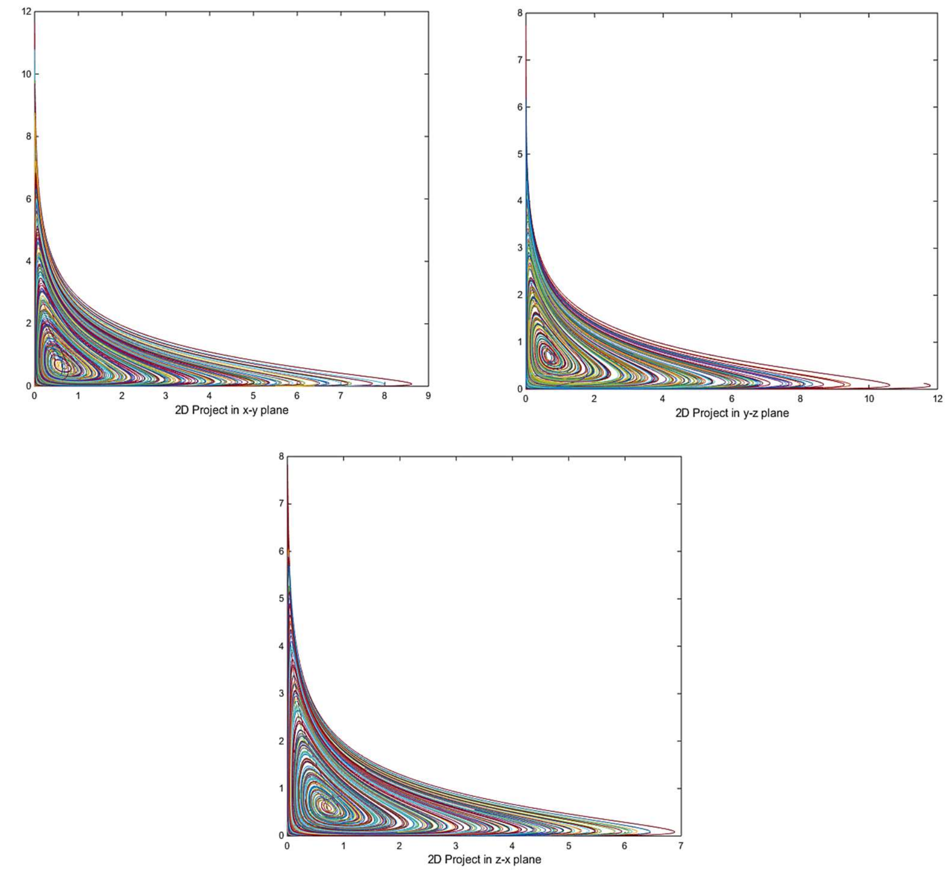


Fig.3.13.5 Projection of chaotic attractor in xy, yz and zx planes for parameter values $\alpha = 0.6658, \beta = 0.9733$ and $\gamma = 0.6277$.

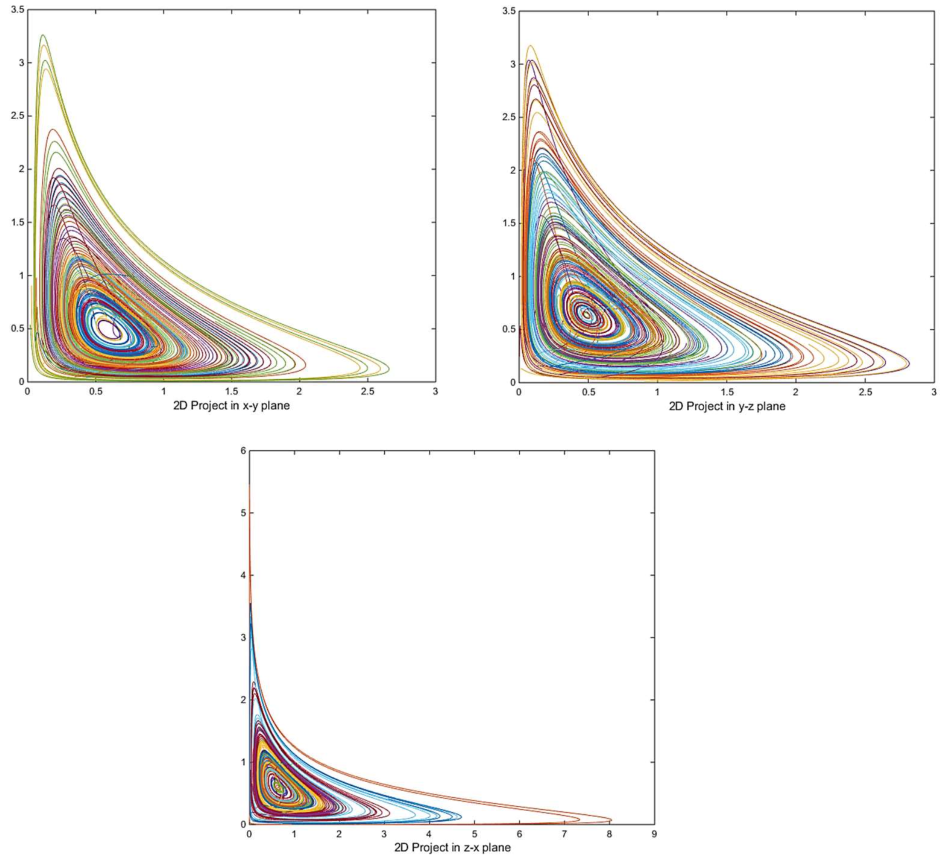


Fig.3.13.6 Projection of chaotic attractor in xy, yz and zx planes for parameter values $\alpha = 0.3440, \beta = 0.5337$ and $\gamma = 0.6278$.

3.5 What happens if all the parameters are the same?

Although the issue about equality of parameters $\alpha, \beta,$ and γ might not attractive, still out of mathematical curiosity, the study aims to understand what happens to the system. Here all the parameters were fixed to be the same. Following we shall have the three-dimensional trajectories with two dimensional projections in the xy, yz and zx planes. The solution of the system was found to remain chaotic.

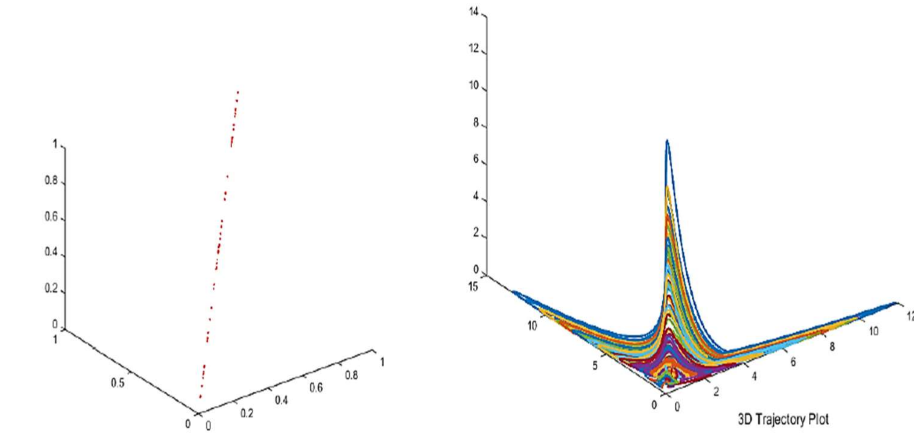


Fig. 3.14. Identical parameters and chaotic trajectory plots.

The chaotic trajectory plots for the parameters corresponding to Fig. 3.13.1 - Fig. 3.13.3 are plotted in the figures Fig. 3.15.1 - Fig. 3.15.3.

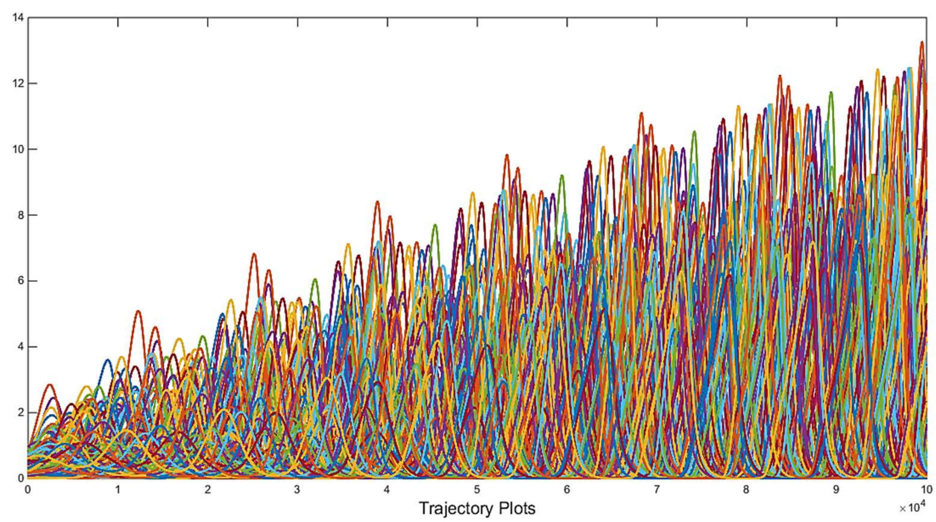


Fig. 3.15.1. Chaotic trajectory plot for parameters $\alpha = 0.2834$, $\beta = 0.6825$, and $\gamma = 0.3581$.

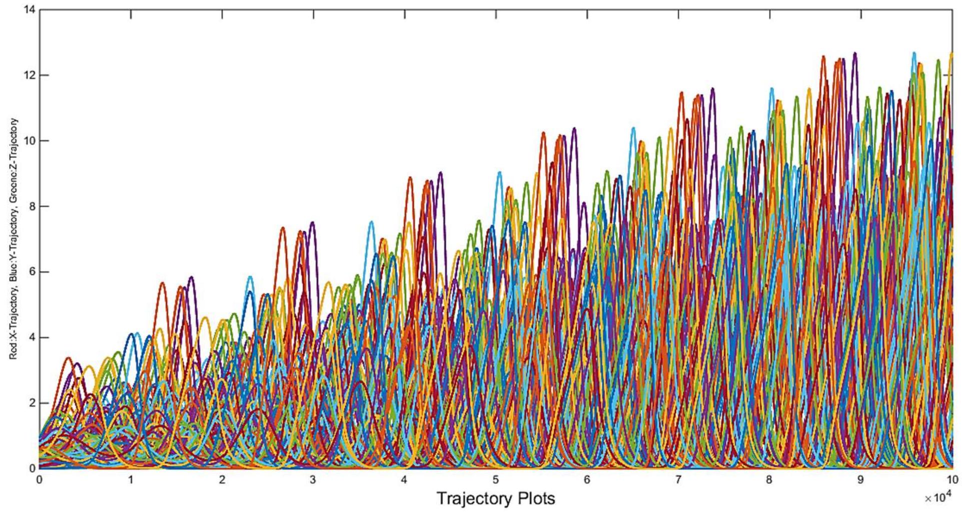


Fig. 3.15.2. Chaotic trajectory plot for parameters $\alpha = 0.6658$, $\beta = 0.9733$, and $\gamma = 0.6277$.

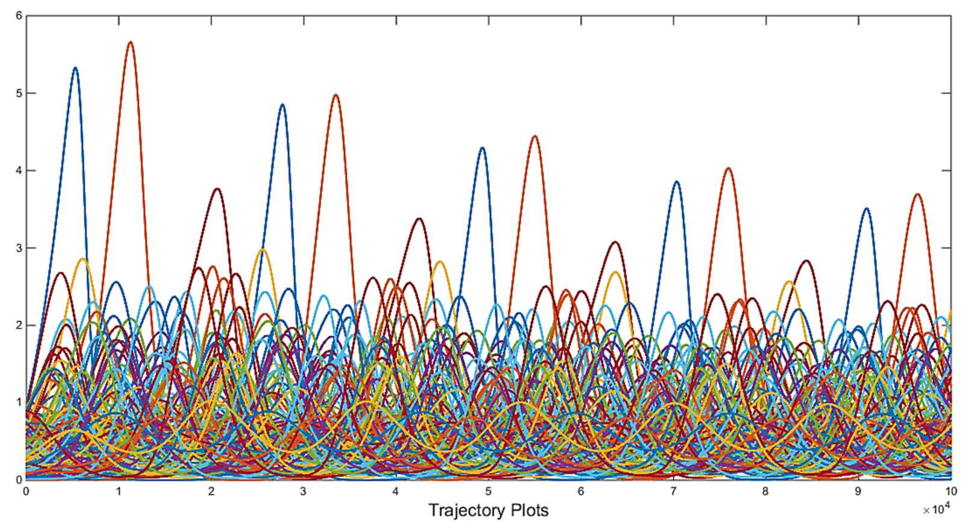


Fig. 3.15.3. Chaotic trajectory plot for parameters $\alpha = 0.3440$, $\beta = 0.5337$, and $\gamma = 0.6278$.

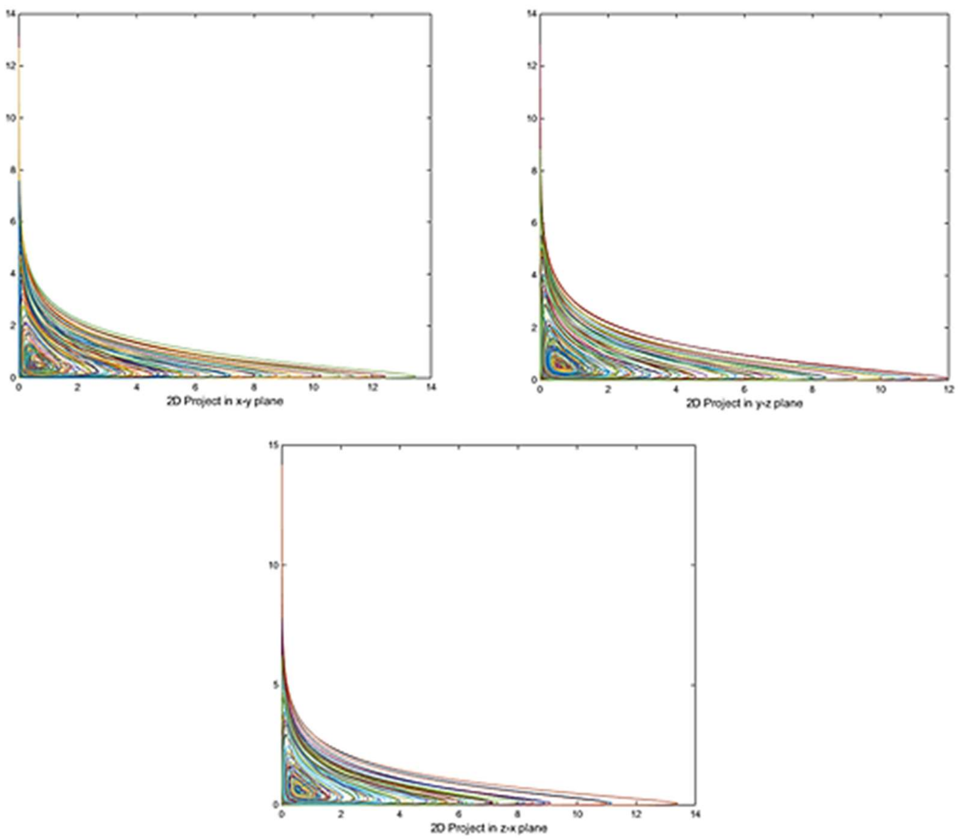


Fig. 3. 16. Projection of Chaotic attractor as shown in Fig. 3. 14 in xy, yz and zx planes.

3.6 Conclusion

In the current chapter, a new three-dimensional chaotic model is defined and analyzed computationally for the growth of cancer cells.

- The model studies the interactions between the tumor cells, healthy tissue cells, and effector immune cells. The competitions among these cells are studied. The model got its inspiration from population dynamics.
- The study establishes the occurrence of chaos for a range of positive parameters $\alpha, \beta,$ and γ.
- The Hurst exponent and fractal dimension are computed for confirming the chaotic dynamics.
- It is observed that all three parameters are very sensitive.

Some previous studies claim that exhibition of chaos by a few similar cancer models but they have not explicitly established the existence of chaos. This work provides an encouragement to develop and study realistic models exhibiting chaos.

CHAPTER 4. COMPUTATIONAL DYNAMICS OF NICHOLSON-BAILEY MODELS

4.1 Introduction

In population dynamics, the Nicholson-Bailey model describes the host-parasitoid system which has been well studied by considering positive real values for all the parameters [17], [31], [44], [90], [91]. In this study, the dynamics in the Nicholson-Bailey model is reinvestigated by considering all the parameters as real numbers and observed that the model has all sorts of dynamics behavior such as chaotic, periodic, and locally stable/unstable equilibriums.

In addition to the classical Nicholson-Bailey system, the dynamics of the scaled and noisy models were considered and observed several interesting results about periodicity and chaotic nature of the models.

A standard basis of population biology models is the host-parasite interactions as presented in [92], [93]. Biological control models played a significant role in the long history of theoretical development. These models focused mainly on the interactions between the hosts and parasitoids. [94].

Understanding the host-parasitoid dynamics is the primary key for biological control. Hence the stability analysis of the host-parasite models has received considerable attention [95].

4.2 The Nicholson-Bailey Model

While investigating the interactions between the hosts and insect parasitoids a discrete model was proposed by Nicholson and Bailey [17]. This model was constructed from the mechanistic viewpoint of searching behavior of parasitoid. The concept is identical to the model of Lotka-Volterra. In this classical model, the parasitoids search independently and randomly in a homogeneous environment, with the consequence that both host and parasitoid populations exhibit diverging oscillations [96]. Later on Hassel and May, assumed that parasites aggregate on host population due to which behavioral stabilization was introduced in the standard Nicholson-Bailey model [18].

The Nicholson-Bailey model is a coupled nonlinear system given by

$$\begin{cases} x_{n+1} = x_n\left(e^{r\left(1-\frac{x_n}{k}\right)-ay_n}\right) \\ y_{n+1} = x_n(1-e^{-y_n}) \end{cases} \qquad \text{Eqn. (4.1)}$$

Populations display lagged oscillations which is a typical nature of the Nicholson-Bailey model [93]. Here peaks and troughs are found in the size of parasite population lagging those in the size of host population until the extinction of parasites. At this point, the population of the hosts explodes.

Hassel had proved that that refuges can help in stabilizing the dynamics of the Nicholson-Bailey model [17]. These refuges sets free some of the hosts from being predated and shield a constant number or proportion of hosts.

Another dynamical system corresponding to the above Nicholson-Bailey model Eqn. (5.1) was defined as follows:

$$\begin{cases} x_{n+1} = (x_n+\alpha)\left(e^{r\left(1-\frac{x_n}{k}\right)-a(y_n+\beta)}\right) \\ y_{n+1} = (x_n+\alpha)(1-e^{-a(y_n+\beta)}) \end{cases} \qquad \text{Eqn. (4.2)}$$

This dynamical system Eqn. (4.2) is a scaled Dynamical system where α and β are scaling factors.

Here x_n and $y_n$are host and parasitoid populations at time n, r is the host reproduction rate, a is the search efficiency and k is the host's carrying capacity [20], [41], [97], [98].

In the present study, these parameters are abstracted and widened over the set of real numbers. Nicholson and Bailey developed realistic models of two species interaction for describing a simple parasitoid-host population dynamics model in discrete time, that became the classical model in population biology [99], [100]. Most of the real world host-parasitoid systems are more stable than Nicholson-Bailey. This shows that the classical model cannot accurately represent the real systems. Therefore some population models with little variants were developed and studied [39], [101]. But in all these models, the parameters of the model were restricted to positive real numbers which is quite natural and make sense [25].

From our curiosity, an attempt has been made to apprehend the general dynamics of the model when the parameters are relaxed over the real line, which is certainly an abstraction of the model. The motivation to extend the parameters towards the real line was to investigate the possibility of bilateral symmetry in dynamics.

Also, the system was studied by introducing Gaussian noise to the Nicholson-Bailey model which enabled us to develop deep understanding of the dynamics of the stochastic model. So, in this work an abstract coupled discrete dynamical system was considered.

4.3 Local Asymptotic Stability of Fixed Points

In this section, both dynamical systems equations $\text{Eqn.}\,(4.1)$ and $\text{Eqn.}\,(4.2)$ were considered to explore the local asymptotic stability of the fixed points [101].

For the system Eqn. (4.1), the fixed points are basically a solution of the system below

$$\begin{cases} \bar{x} = \bar{x}\left(e^{r\left(1-\frac{\bar{x}}{k}\right)-a\bar{y}}\right) \\ \bar{y} = \bar{x}(1-e^{-a\bar{y}}) \end{cases} \qquad \text{Eqn. (4.3)}$$

Here $(0,0)$ is trivial fixed point and one of the other non-trivial fixed points is $(k,0)$. The stability of these two fixed points, which is locally asymptotic, is explored as follows:

For the fixed point $(\bar{x}, \bar{y})$, the local stability depends on the character of the eigenvalues of the Jacobian at $(\bar{x}, \bar{y})$. The Jacobian about the fixed point $(\bar{x}, \bar{y})$ is denoted as $J_{(\bar{x},\bar{y})}$. Here,

$$J_{(\bar{x},\bar{y})} = \begin{pmatrix} e^{r\left(1-\frac{\bar{x}}{k}\right)-a\bar{y}} - \frac{e^{r\left(1-\frac{\bar{x}}{k}\right)-a\bar{y}} r\bar{x}}{k} & 1 - e^{-a\bar{y}} \\ -ae^{r\left(1-\frac{\bar{x}}{k}\right)-a\bar{y}}\bar{x} & ae^{-a\bar{y}}\bar{x} \end{pmatrix} \qquad \text{Eqn. (4.4)}$$

In concluding the nature of the fixed points, the following classical theorem is very important.

Theorem 4.1 : The fixed point $(\bar{x}, \bar{y})$ is an attractor / repeller if the modulus of each eigenvalue of the Jacobian $J_{(\bar{x},\bar{y})}$ is less than 1/greater than 1 respectively. If the moduli of the eigenvalues are 1, $(\bar{x}, \bar{y})$ is a saddle point.

The eigenvalues of $J_{(0,0)}$ are obtained as 0 and e^r. Hence by the above theorem 4.1, the fixed point (0,0)

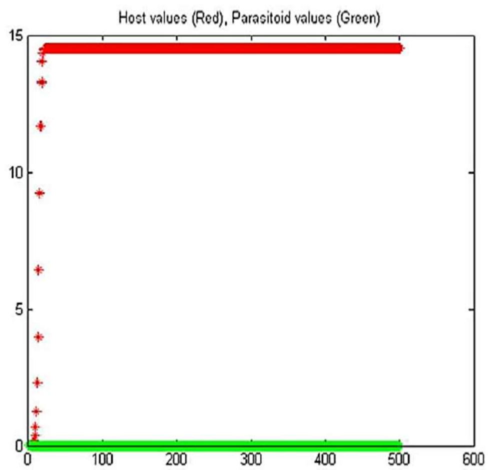
- becomes an attractor, if $r < 0$.
- becomes a saddle point, if $r = 0$.
- becomes a repeller, if $r > 0$.

About the fixed point $(k, 0)$, the eigenvalues are ak and $1 - r$ and therefore by the above theorem, the fixed point $(k, 0)$

- is attracting if $\frac{1}{k} < a < \frac{-1}{k}$ while $k < 0$ *and* $0 < r < 2$.
- is attracting if $\frac{-1}{k} < a < \frac{1}{k}$ while $k > 0$ *and*$0 < r < 2$.
- is repelling if $a < \frac{1}{k}$ *or* $a > \frac{-1}{k}$ while $k < 0$ *and* $r < 0$ *or* $r > 2$.
- is repelling if $a < \frac{-1}{k}$ *or* $a > \frac{1}{k}$ while $k > 0$ *and* $r < 0$ *or* $r > 2$.
- is a saddle point if $= \frac{\pm 1}{k}$ *and* $r = 0$ *or* $r = 2$.

Here demonstrate two examples concentrating about the fixed points (0,0) and $(k, 0)$.

Example 4.1: Consider $r = 0.119 > 0, a = 0.2 \; and \; k = 14.49$, the point (0,0) is a repeller and in this case the trajectory near the neighborhood of (0,0) with initial value $(0.001, 0.005)$ converges to $(k, 0)$ as shown in Figure 4.1: Left. While $r = 0, a = 0.2, and \; k = 14.49$, the fixed point (0,0) is a saddle point. In this case, the trajectory near the neighborhood of (0,0) with an initial value $(0.001, 0.005)$ converges to (0.00099900300093254,0) which is shown in Figure 4.1: Right. The trajectory plots of 500 iterations with initial value (0.001, 0.005) have been figured in Fig 4.1.



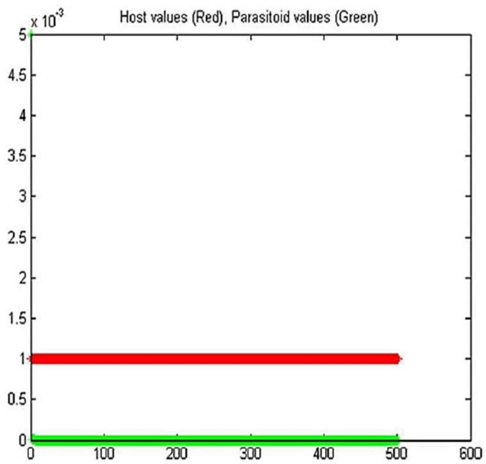


Fig 4. 1: Trajectory plots: Left Repeller; Right: Saddle point. x-axis and y-axis denote the number of iterations and trajectories respectively.

Example 4.2: Considering $r = 1.55 \in (0,2), a = 0.05, and \; k = 16$, the point $(k, 0)$ is attracting and in this case the trajectory near the neighborhood of $(k, 0)$ with an initial value $(15.799, 0.0011)$ converges to $(k, 0)$ (Fig 4.2 (top)). While $r = 2.55, a = 0.05, and \; k = 16$, the point $(k, 0)$ is repelling and in this case the trajectory near the neighborhood of $(k, 0)$ with an initial value (15, 0.001) possessed a period 4 attracting cycle (25.7440, 0), (5.448, 0), (29.282,0), (3.526,0) (Fig 4.2 (bottom).

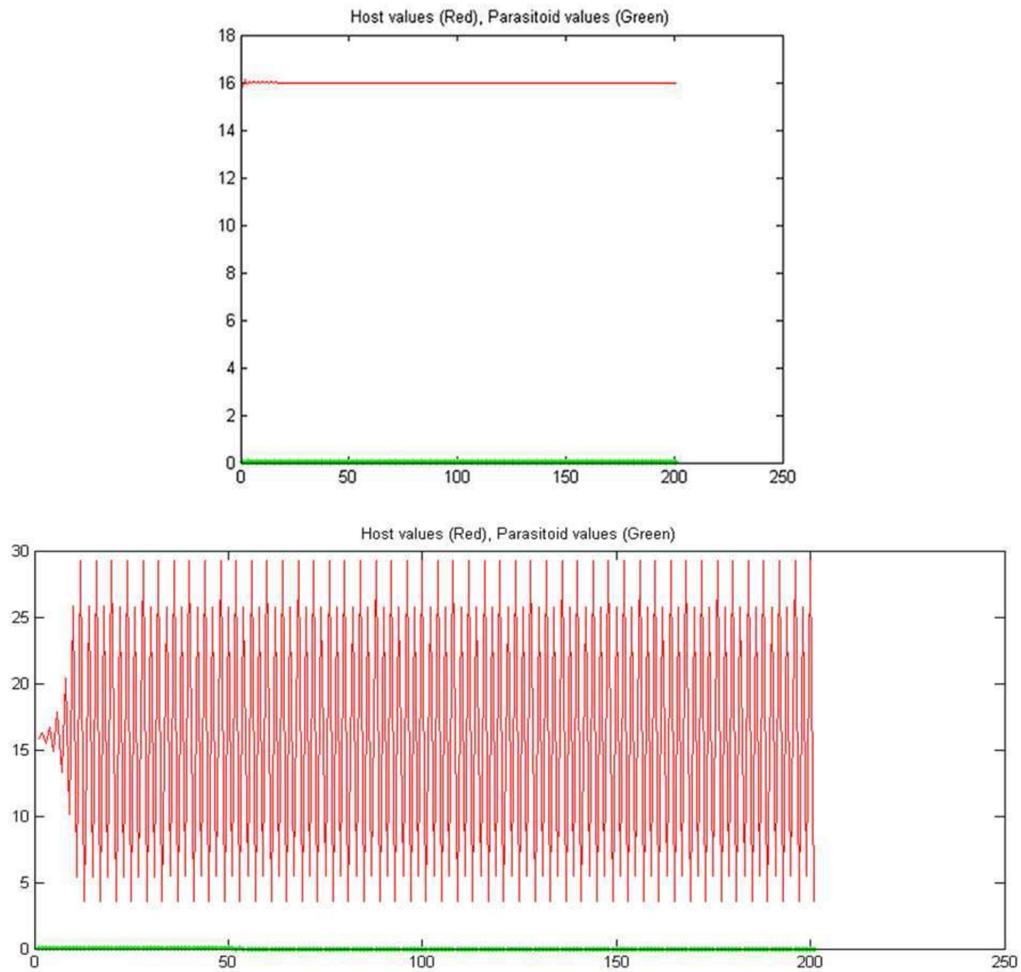


Fig 4. 2: Trajectory plots (top: attracting $(k,0)$; bottom: repeller $(k,0)$).

When $r = 2.2, a = 0.05$ and $k = -16$, the fixed point $(k, 0)$ is repelling and in this case the trajectory near the neighborhood of $(k, 0)$ with initial value (-15.799, 0.0011) possesses a period 2 cycle (-24.0470,0), (-7.9530,0) as shown in Fig 4.3 (top). If $r = 2, a = 0.0625, and\ k = -16$, the point $(k, 0)$ is a saddle point and in this case the trajectory near the neighborhood of $(k, 0)$ with initial value (-15.799,0.0011) approaches to the saddle point (Fig 4.3 (bottom)).

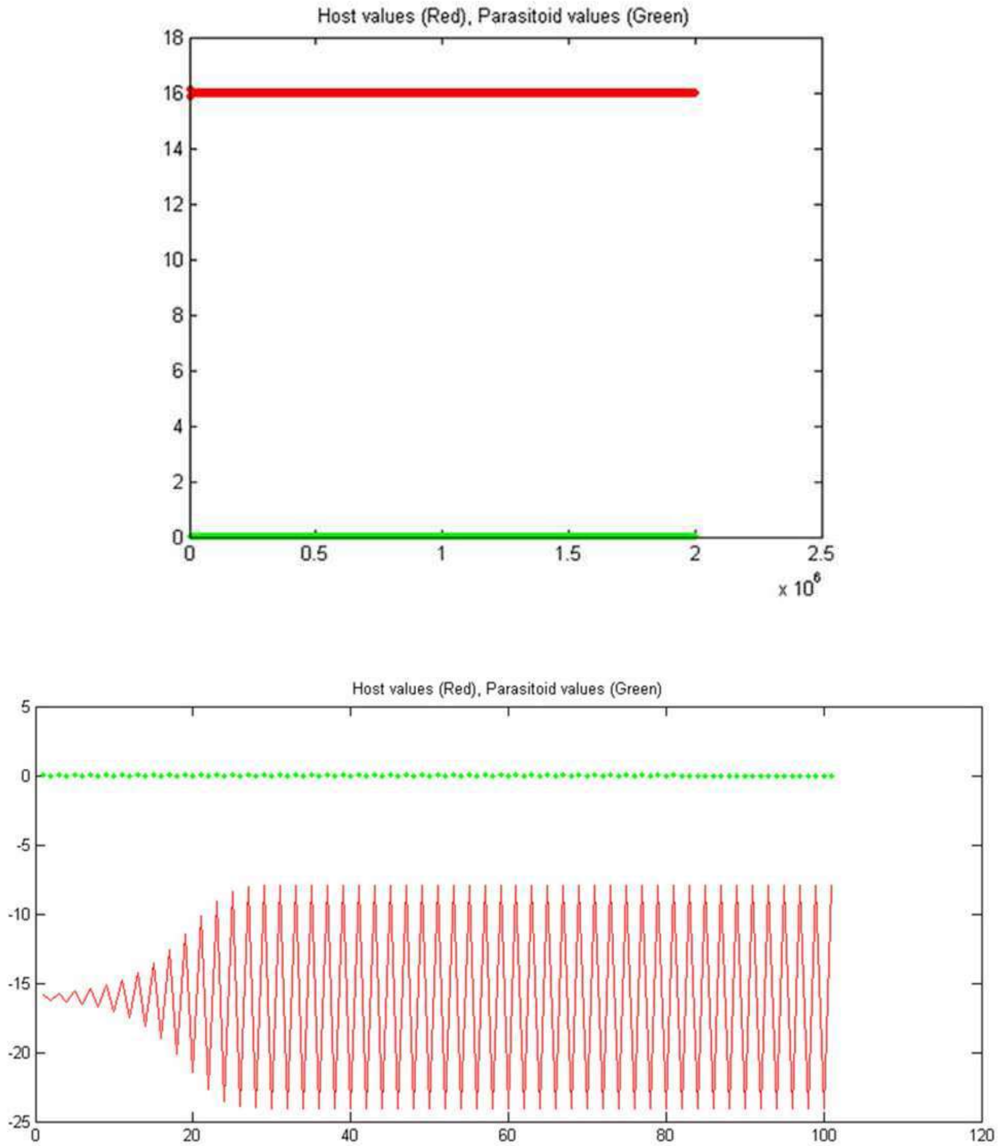


Fig 4.3: Trajectory plots (top: saddle $(k, 0)$ and bottom: repeller $(k, 0)$).

We tried to search the space parameters (a, k, r) for which the fixed point $(0,0)$ is attracting, repelling or saddle. By computer simulation, obtained the following space of parameters (Fig.4.4) of three different kinds. Also, obtained the space of parameters (a, k, r) such that the fixed point $(k, 0)$ is attracting, repelling or saddle (Fig.4.5).

We shall now find out the parameter space where one of the fixed points is attracting (repelling) and the other one is repelling (attracting). The plot is given in Fig.4.6. It was observed that there does not exist any parameter (a, k, r) such that both the fixed points are either attracting or repelling.

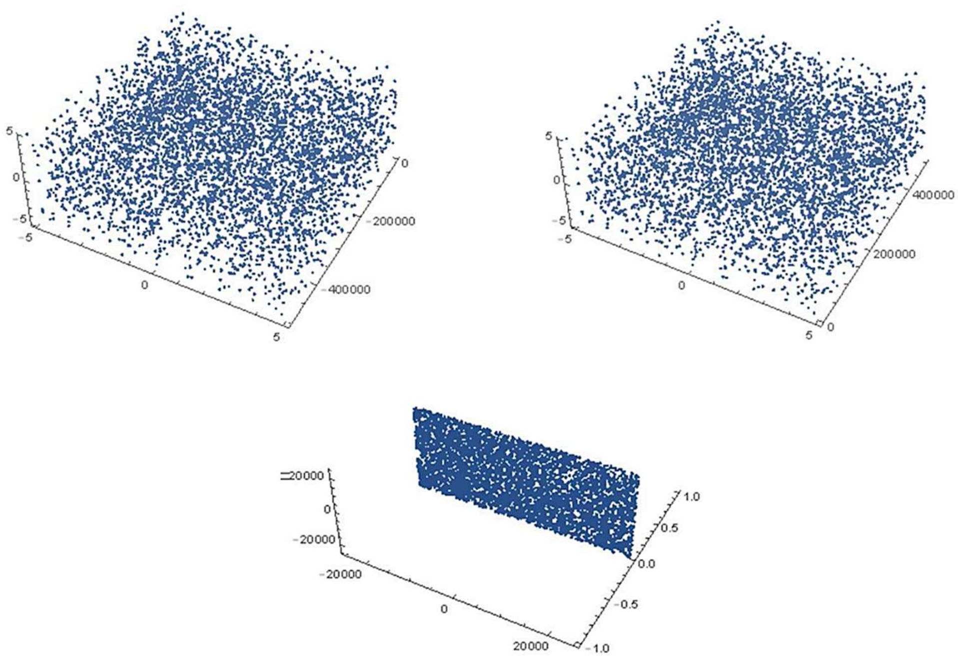


Fig.4.4. Top Left: Attracting $(\mathbf{0}, \mathbf{0})$. Top Right : Repeller $(\mathbf{0}, \mathbf{0})$. Bottom: Saddle $(\mathbf{0}, \mathbf{0})$. Here in 3D coordinate system $\boldsymbol{x}-, \boldsymbol{y}-, \boldsymbol{z}-\boldsymbol{axes}$ denote the parameters $\boldsymbol{a}, \boldsymbol{k}, \boldsymbol{r}$ respectively.

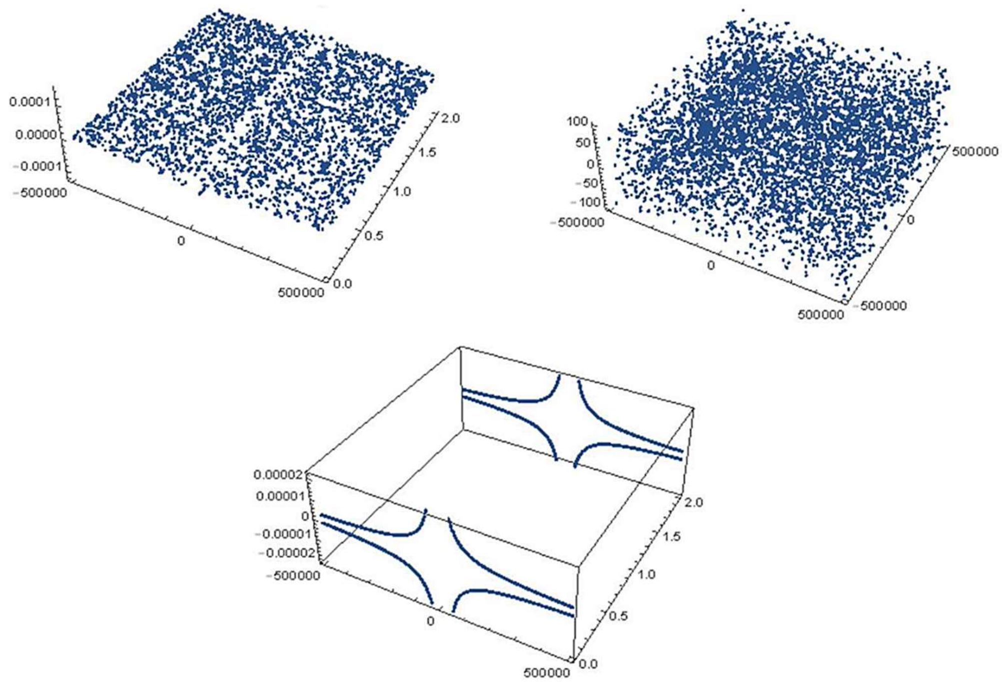


Fig.4.5. Top Left: Attracting $(k, 0)$, Top right: Repeller $(k, 0)$ and bottom: Saddle $(k, 0)$.

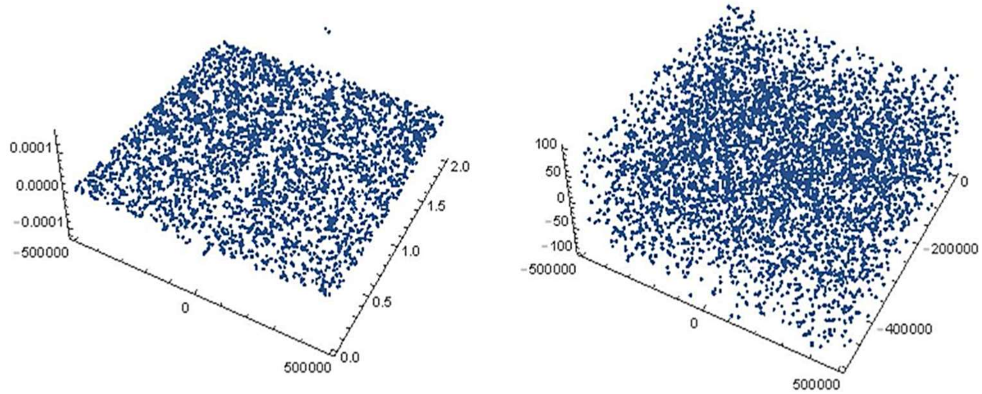


Fig.4.6. Left: $(0, 0)$ is repelling and $(k, 0)$ is attracting. Right: $(0, 0)$ is attracting and $(k, 0)$ is repelling.

The following Table 4.1 lists the set of parameters $a, k\ and\ r$ of the Nicholson-Bailey mode Eqn. (4. 1) such that the point (0,0),((k, 0)) is attracting but the point (k, 0) ((0,0)) is repelling even if we consider the initial conditions in the neighborhood of the points (k, 0) ((0,0)).

Table 4.1: Set of parameters when $(\mathbf{0}, \mathbf{0})$ is attracting $(\boldsymbol{k}, \mathbf{0})$ is repelling and $(\boldsymbol{k}, \mathbf{0})$ is attracting with $(\mathbf{0}, \mathbf{0})$ repelling.

Parameter (a, k, r)	Initial value	Remark
$(0.03, 11, 0.25)$	$x_0 = 0.6781, y_0 = 0.2135$	$(k, 0)$ is attracting where (0,0)is repelling.
$(12.27,28.45, -7.72)$	$x_0 = 28.7, y_0 = 0.02$	(0,0) is attracting where $(k, 0)$is repelling.
$(1, 19, 0.59)$	$x_0 = 0.012, y_0 = 0.231$	$(k, 0)$ is attracting where (0,0)is repelling
$(42.33, 27, -48.89)$	$x_0 = 27.3, y_0 = 0.02$	(0,0) is attracting where $(k, 0)$is repelling
$(0.6, 19, 0.83)$	$x_0 = 0.0046, y_0 = 0.7749$	$(k, 0)$ is attracting where (0,0)is repelling

It is noted that when the fixed point (0,0) is attracting then the fixed point $(k, 0)$ is not able to attract even if the initial value is chosen around the non-trivial fixed point $(k, 0)$. This observation is also true for the fixed point $(k, 0)$. The fact has been justified through a couple of examples as stated in Table 4. 1. This is because of the fact that when $|1 - r|$ are greater than 1 then $|r| > 1$. It is needless to mention that $ak, 1 - r$ are eigenvalues corresponding to $(k, 0)$ and e^r and 0 are the eigenvalues of $J_{(0,0)}$ of the system Eqn. (4.1).

So far we did find (0,0) and $(k, 0)$ are fixed points of the system Eqn. (4.1) and characterized them. In this regard, one of the most relevant results is stated as below.

Theorem 4.2: If $(\bar{x}, \bar{y})$ is a fixed point of the Nicholson Bailey system Eqn.(4.1) with the parameters (a, k, r) then $(-\bar{x}, -\bar{y})$ is also a fixed point of the Nicholson-Bailey system Eqn. (4.1) with the parameters $(-a, r, -k)$.

Proof. Since $(\bar{x}, \bar{y})$ is a fixed point of the Nicholson-Bailey system Eqn. (4.1) with the parameters (a, k, r) then we have

$$\bar{x} = \bar{x}\left(e^{r\left(1-\frac{\bar{x}}{k}\right)-a\bar{y}}\right), \quad \bar{y} = \bar{x}(1 - e^{-a\bar{y}}) \qquad \text{Eqs. } (4.5)$$

Rewriting Eqs. (4.5) as

$$-\bar{x} = -\bar{x}\left(e^{r\left(1-\frac{-\bar{x}}{-k}\right)-(-a)(-\bar{y})}\right), -\bar{y} = -\bar{x}\left(1 - e^{-(-a)(-\bar{y})}\right) \qquad \text{Eqs. } (4.6)$$

From Eqn. (4.6) it is seen that $(-\bar{x}, -\bar{y})$ is also a fixed point of the Nicholson-Bailey system Eqn. (4.1) with the parameters $(-a, r, -k)$.

Theorem 4. 3: If $(\bar{x}, \bar{y})$ is a fixed point of the Nicholson Bailey system Eqn. (4.1) with the parameters (a, k, r) then $(c\bar{x}, c\bar{y})$ is also a fixed point of the Nicholson-Bailey system Eqn. (4.1) with the parameters $(\frac{a}{c}, r. ck)$, where *c* is any real number.

Proof. Since $(\bar{x}, \bar{y})$ is a fixed point of the Nicholson-Bailey system Eqn. (4.1) with the parameters (a, k, r) then we have

$$\bar{x} = \bar{x}\left(e^{r\left(1-\frac{\bar{x}}{k}\right)-a\bar{y}}\right), \bar{y} = \bar{x}(1 - e^{-a\bar{y}}) \qquad \text{Eqs. } (4.7)$$

Rewriting Eqs. (4.7) as

$$c\bar{x} = c\bar{x}\left(e^{r\left(1-\frac{c\bar{x}}{ck}\right)-(\frac{a}{c})(c\bar{y})}\right), c\bar{y} = c\bar{x}\left(1 - e^{-(\frac{a}{c})(c\bar{y})}\right) \qquad \text{Eqs. } (4.8)$$

From Eqs. (4.8) it is seen that $(c\bar{x}, c\bar{y})$ is also a fixed point of the Nicholson-Bailey system Eqn. (4.1) with the parameters $(\frac{a}{c}, r. ck)$, where *c* is any real number.

Remark 4. 1. The local asymptotic behavior of the fixed point $(c\bar{x}, c\bar{y})$ of the Nicholson-Bailey system Eqn. (4.1) is same as the local asymptotic behavior of the fixed point $(\bar{x}, \bar{y})$.

We now try to see if there are any other fixed points of Eqn (4.1). By computer simulation, obtained a set of 5000 different fixed points for different sets of system parameters. The plot of those fixed points including the corresponding parameters a, k and r is given in Fig. 4.7.

It is evident from Fig. 4.7 that the fixed points of system Eqn. (4.1) is symmetric about the line $y = x$ where the parameters a, k and r lie in the interval $(-6,6)$, $(-5 \text{ x } 10^6, 5 \text{ x } 10^6)$ and $(-4,4)$ respectively. It is observed that the parameter k is inversely related to parameter a. From the simulation, it is observed that there does not exist any fixed points in the region generated by the anti-clockwise rotation of the line $y = x \tan\theta$, where θ lies in the interval $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$.

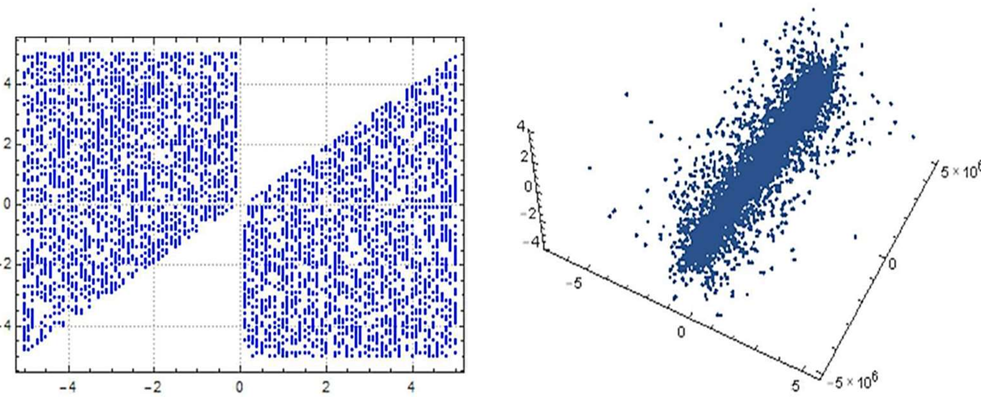


Fig.4.7. Plot of fixed points and corresponding parameters (a, k, r) of the Nicholson-Bailey system Eqn. (4.1)

4.4 Local asymptotic stability of the scaled Nicholson-Bailey model

The fixed points of the system Eqn (4.2) are basically a solution of the following system.

$$\bar{x} = (\bar{x}+\alpha)\left(e^{r\left(1-\frac{(\bar{x}+\alpha)}{k}\right)-a(\bar{y}+\beta)}\right), \quad \bar{y} = (\bar{x}+\alpha)\left(1-e^{-a(\bar{y}+\beta)}\right) \quad \text{Eqs. (4.9)}$$

It was obtained that the system Eqn. (4.1) possess two fixed points $(0,0)$ and $(k, 0)$. Further, investigated if there are any parameters r, a and k such that $(0,0)$ and $(k, 0)$ remain fixed points of the scaled system Eqn. (4.2). It is found that if $\alpha = 0$, and β is any real number, then $(0,0)$ remains a fixed point. Also there are parameters r, a and k in the scaled system Eqn. (4.2) for some α and β for which the point $(k, 0)$ is a fixed point.

If both α and β are non-zero real numbers, the other possible fixed points are $(\alpha, 0), (\alpha, \alpha), (\beta, 0), (\beta, \beta)$ and (α, β) for certain parameters r, a and k. We have analogous theorems (Theorem 4.4-4.5) corresponding to (Theorem 4.2-4.3) for scaled dynamical system Eqn. (4.2).

Theorem 4. 4: If $(\bar{x}, \bar{y})$ is a fixed point of the Eqn. (4.2) with parameters (a, k, r) then $(-\bar{x}, -\bar{y})$ is also a fixed point to the Nicholson-Bailey system Eqn. (4.2) with the parameters $(-a, r, -k)$ with the scaling factor $(-\alpha, -\beta)$.

Proof: Since $(\bar{x}, \bar{y})$ is a fixed point of the Nicholson-Bailey system Eqn. (4.2) with the parameters (a, k, r) and scaling factor (α, β), then we have

$$\bar{x} = (\bar{x} + \alpha)\left(e^{r\left(1 - \frac{(\bar{x}+\alpha)}{k}\right) - a(\bar{y}+\beta)}\right), \ \bar{y} = (\bar{x} + \alpha)(1 - e^{-a(\bar{y}+\beta)}) \quad \text{Eqs. (4.10)}$$

Rewriting Eqs (4.10) as

$$\begin{cases} -\bar{x} = \left(-\bar{x} + (-\alpha)\right)\left(e^{r\left(1 - \frac{\left(-\bar{x}+(-\alpha)\right)}{k}\right) - a\left(-\bar{y}+(-\beta)\right)}\right) \\ \qquad -\bar{y} = (-\bar{x} + (-\alpha))(1 - e^{-a(-\bar{y}+(-\beta))}) \end{cases} \quad \text{Eqs. (4.11)}$$

From Eqs. (4.11) it is seen that $(-\bar{x}, -\bar{y})$ is also a fixed point of the Nicholson-Bailey system Eqn. (4.2) with the parameters $(-a, r, -k)$ and scaling factor $(-\alpha, -\beta)$.

Theorem 4. 5: If $(\bar{x}, \bar{y})$ is a fixed point of the Nicholson Bailey system Eqn. (4.2) with the parameters (a, k, r) then $(c\bar{x}, c\bar{y})$ is also a fixed point of the Nicholson-Bailey system Eqn. (4.2) with the parameters $(\frac{a}{c}, r. ck)$, where *c* is any real number and scaling factor $(\frac{\alpha}{c}, \frac{\beta}{c})$.

Proof: Since $(\bar{x}, \bar{y})$ is a fixed point of the Nicholson-Bailey system Eqn. (4.2) with the parameters (a, k, r) and scaling factor (α, β), then we have

$$\bar{x} = (\bar{x} + \alpha)\left(e^{r\left(1 - \frac{(\bar{x}+\alpha)}{k}\right) - a(\bar{y}+\beta)}\right), \bar{y} = (\bar{x} + \alpha)(1 - e^{-a(\bar{y}+\beta)}) \quad \text{Eqs. (4.12)}$$

Rewriting Eqs. (4.12) as

$$\begin{cases} c\bar{x} = \left(c\bar{x} + c\left(\frac{\alpha}{c}\right)\right)\left(e^{r\left(1-\frac{\left(c\bar{x}+c\left(\frac{\alpha}{c}\right)\right)}{ck}\right)-\left(\frac{a}{c}\right)\left(c\bar{y}+c\left(\frac{\beta}{c}\right)\right)}\right) \\ c\bar{y} = (c\bar{x} + c(\frac{\alpha}{c}))(1 - e^{-(\frac{a}{c})(c\bar{y}+c(\frac{\beta}{c}))}) \end{cases} \qquad \text{Eqs. } (4.13)$$

From Eqs. (4.13) it is seen that $(c\bar{x}, c\bar{y})$ is also a fixed point of the Nicholson-Bailey system Eqn. (4. 2) with the parameters $(\frac{a}{c}, r, ck)$ and scaling factor $(\frac{\alpha}{c}, \frac{\beta}{c})$.

In addition, through a computer simulation, found the set of fixed points which is depicted in Fig. 4. 8.

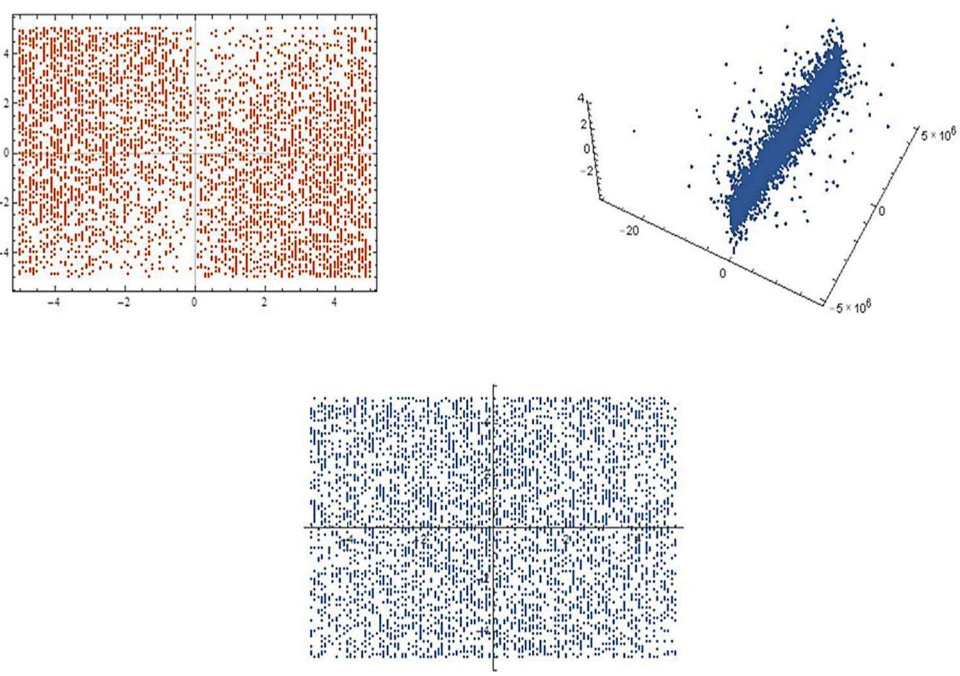


Fig.4.8. Plot of fixed points and corresponding parameters (a, k, r) and the scaling factors (α, β) of the scaled Nicholson-Bailey system Eqn. (4.2)

In the scaled system Eqn. (4.2), the fixed points range over the two dimensional plane unlike in the case of the original Nicholson-Bailey system Eqn. (4.1). Also, the parameters are considered to be ranged over the plane. Therefore, it can be concluded that fixed points set of the Nicholson-Bailey system Eqn. (4.1) is a subset of the scaled system Eqn. (4.2) fixed points.

For the fixed points $(\bar{x}, \bar{y})$ of the scaled system Eqn. (4.2), the local stability depends on the character of the eigenvalues of the Jacobian at $(\bar{x}, \bar{y})$. About $(\bar{x}, \bar{y})$, the Jacobian is denoted as $J_{(\bar{x},\bar{y})}$ given by

$$J_{(\bar{x},\bar{y})} = \begin{pmatrix} e^{r\left(1-\frac{\bar{x}+\alpha}{k}\right)-a(\bar{y}+\beta)} - \dfrac{e^{r\left(1-\frac{\bar{x}+\alpha}{k}\right)-a(\bar{y}+\beta)} r(\bar{x}+\alpha)}{k} & 1-e^{-a(\bar{y}+\beta)} \\ -ae^{r\left(1-\frac{(\bar{x}+\alpha)}{k}\right)-a(\bar{y}+\beta)}(\bar{x}+\alpha) & ae^{-a(\bar{y}+\beta)}(\bar{x}+\alpha) \end{pmatrix}$$

The local asymptotic stability of couple of fixed points of Eqn. (4.2) are studied.

4.4.1 Stability of Fixed Point $(\mathbf{0},\mathbf{0})$

The fixed point (0,0) exists for a range of parameters a, k, and r. To establish the existence of the fixed point (0,0), presented the figure of parameter space (a, r, k) as well as the scaling factor space (α, β) in Fig.4. 9. We can notice that the point (0,0) is a fixed point only if $\alpha = 0$ and β is any real number which is evident from Fig 4. 9.

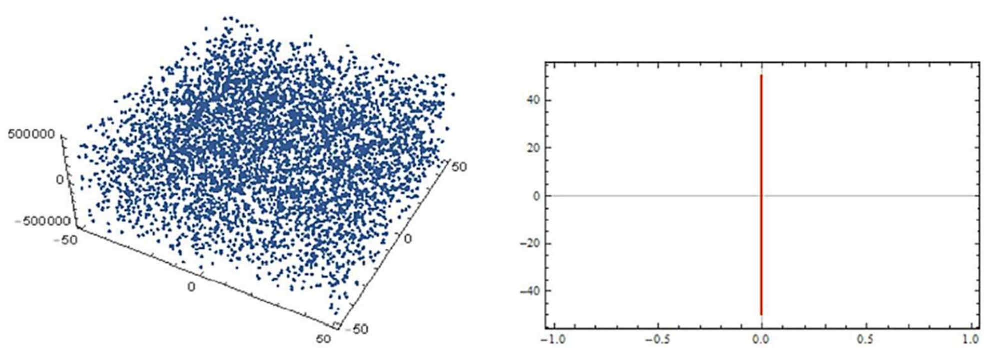


Fig.4.9: Plot of parameter space (a, k, r) and scaling factors (α, β) of the Nicholson-Bailey system (4.2) where (0,0) is a fixed point.

The eigenvalues of the $J_{(0,0)}$ are

$$e^{-\alpha}\left(a\alpha k-\sqrt{\left(-a\alpha k+\alpha re^{r\left(1-\frac{\alpha}{k}\right)}-ke^{r\left(\left(1-\frac{\alpha}{k}\right)\right)}\right)^{2}-4\left(a\alpha k^{2}e^{\alpha\beta+r\left(1-\frac{\alpha}{k}\right)}-a\alpha^{2}kre^{r\left(1-\frac{\alpha}{k}\right)}\right)}\right.$$
$$\left.+\alpha r\left(-e^{r\left(1-\frac{\alpha}{k}\right)}\right)+ke^{r\left(1-\frac{\alpha}{k}\right)}\right)/\,2k$$

and

$$e^{-\alpha\beta}\left(a\alpha k+\sqrt{\left(-a\alpha k+\alpha re^{r\left(1-\frac{\alpha}{k}\right)}-ke^{r\left(\left(1-\frac{\alpha}{k}\right)\right)}\right)-4\left(a\alpha k^{2}e^{\alpha\beta+r\left(1-\frac{\alpha}{k}\right)}-a\alpha^{2}kre^{r\left(1-\frac{\alpha}{k}\right)}\right)^{2}}\right.$$
$$\left.+\alpha r\left(-e^{r\left(1-\frac{\alpha}{k}\right)}\right)+ke^{r\left(1-\frac{\alpha}{k}\right)}\right)/\,2k$$

Here an example was presented with set of parameters where the fixed point (0,0) is attracting.

Example 4. 3: consider $\alpha = 0$ and β as non zero scaling factor. The moduli of both eigenvalues are less than 1 for a set of values of parameters $r = 0.5, a, k =$ $\{-236.7, -49\}, \{-155.1, -183.3\}, \{69.4, -175.7\},$ $\{-242.7, -18.7\}, \{26, -72.7\}, \{245.4, 28.7\}, \{-55.9, 36.6\},$ $\{-108, 137.3\}, \{51.6, 61.1\}, \{-54.7, -49.8\}$ and corresponding scaling factor $\beta: \{-180.7, -8.6, 9.9, -176.7, 152.1, 148.4, -214.1, -172.7, 124, -38.8\}$ respectively. In this case, the fixed point (0,0) is locally asymptotically stable (attracting). It was found that there does not exist any parameter a, k, and r such that the fixed point (0,0) is repelling.

4.4.2 Stability of fixed point $(\alpha, 0)$

In establishing the existence of the fixed point $(\alpha, 0)$, a computer simulation is done and found the space of parameters (a, r, k) with 5000 points as well as the scaling factors space (α, β) which is depicted in Fig. 4.10.

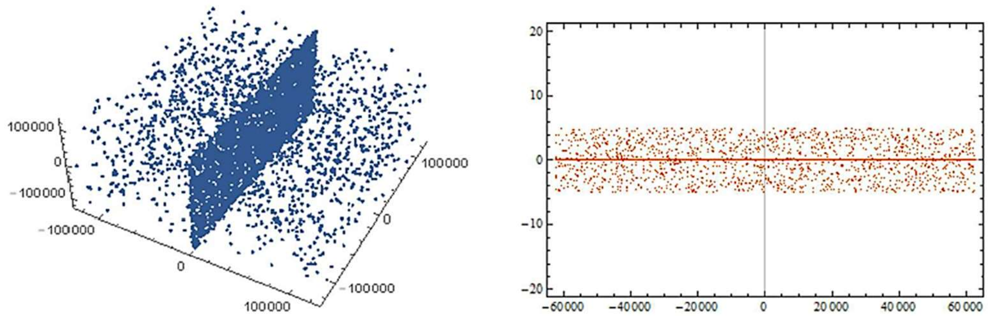


Fig. 4.10. Plot of parameter space $(\boldsymbol{a}, \boldsymbol{r}, \boldsymbol{k})$ and the scaling factor $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ of the scaled system Eqn. $(\mathbf{4.2})$ where $(\boldsymbol{\alpha}, \mathbf{0})$ is a fixed point.

The eigenvalues of $J_{(\alpha,0)}$ are

$$\left(e^{-\alpha\beta}\left(-\sqrt{\left(-2a\alpha k+2\alpha re^{r\left(1-\frac{2\alpha}{k}\right)}-ke^{r\left(\left(1-\frac{2\alpha}{k}\right)\right)}\right)^2-4\left(2a\alpha k^2e^{a\beta+r\left(1-\frac{2\alpha}{k}\right)}-4a\alpha^2kre^{r\left(1-\frac{2\alpha}{k}\right)}\right)}+2\alpha r\left(-e^{r\left(1-\frac{2\alpha}{k}\right)}\right)+ke^{r\left(1-\frac{2\alpha}{k}\right)}+2a\alpha k\right)\right)/\,2k$$

and

$$\left(e^{-\alpha\beta}\left(\sqrt{\left(-2a\alpha k+2\alpha re^{r\left(1-\frac{2\alpha}{k}\right)}-ke^{r\left(\left(1-\frac{2\alpha}{k}\right)\right)}\right)^2-4\left(2a\alpha k^2e^{a\beta+r\left(1-\frac{2\alpha}{k}\right)}-4a\alpha^2kre^{r\left(1-\frac{2\alpha}{k}\right)}\right)}+2\alpha r\left(-e^{r\left(1-\frac{2\alpha}{k}\right)}\right)+ke^{r\left(1-\frac{2\alpha}{k}\right)}\right)+2a\alpha k\right)/2k$$

Here are two examples of sets of parameters where the fixed point $(\alpha, 0)$ is attracting and repelling.

Example 4. 4: consider $\alpha = 3$ and $\beta = 10$ as scaling factors for the scaled system (4.2). The moduli of both eigenvalues are less than 1 for a set of values of $r = 2.5, a, k = \{110.5,100.5\}, \{43.7, -149\}, \{91.1, -7.2\}, \{69.4, -175.7\}$ $, \{42.3, -57.5\}, \{195.5,45.1\}, \{144.3,34\}, \{131.8,135\}, \{213.9, -64.5\}$.

Here, the fixed point (3,0) is locally asymptotically stable (attracting).

Example 4. 5: consider $\alpha = 5$ and $\beta = 8$ as scaling factors for the scaled system Eqn. (4.2). The moduli of both eigenvalues are greater than 1 for a set of values of $r = 2.5, a, k = \{-157.4, -38.9\}, \{-227, -215.7\}$, $\{-158, -207.5\}, \{-193.6, 104.4\}, \{-86.4, -159.2\}, \{-234.4, 202\}$, $\{-108, 137.3\}, \{-179.8, -93.7\}, \{-110.9, -98.4\}, \{-155.1, -183.3\}$.

Here, the fixed point (5,0) is repelling.

4.4.3 Stability of fixed point $(\mathbf{0}, \boldsymbol{\alpha})$

In establishing the existence of the fixed point $(0, \alpha)$, a computer simulation is done and found the space of parameters (a, r, k) with 5000 points as well as the scaling factors space (α, β) which is depicted in Fig. 4.11.

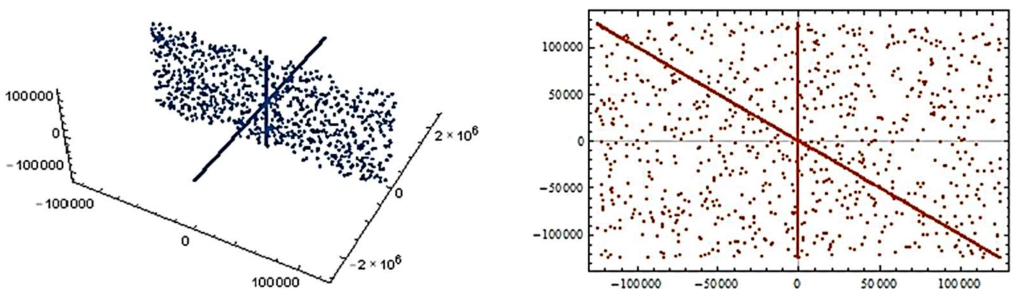


Fig.4.11. Plot of parameter space $(\boldsymbol{a}, \boldsymbol{r}, \boldsymbol{k})$ and the scaling factor $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ of the scaled system Eqn. $(\mathbf{4.2})$ where $(\mathbf{0}, \boldsymbol{\alpha})$ is a fixed point.

The eigenvalues of $J_{(0,\alpha)}$ are

$$\left(e^{-(\alpha+\beta)}\left(-\sqrt{\left(-a\alpha k+\alpha r e^{r\left(1-\frac{\alpha}{k}\right)}-k e^{r\left(\left(1-\frac{\alpha}{k}\right)\right)}\right)^{2}-4\left(a\alpha k^{2} e^{(\alpha+\beta)+r\left(1-\frac{\alpha}{k}\right)}-a\alpha^{2} k r e^{r\left(1-\frac{\alpha}{k}\right)}\right)}+\alpha r\left(-e^{r\left(1-\frac{\alpha}{k}\right)}\right)+k e^{r\left(1-\frac{\alpha}{k}\right)}\right)+a\alpha k\right)/2k$$

and

$$\left(e^{-(\alpha+\beta)}\left(\sqrt{\left(-a\alpha k+\alpha r e^{r\left(1-\frac{\alpha}{k}\right)}-k e^{r\left(\left(1-\frac{\alpha}{k}\right)\right)}\right)^{2}-4\left(a\alpha k^{2} e^{(\alpha+\beta)+r\left(1-\frac{\alpha}{k}\right)}-a\alpha^{2} k r e^{r\left(1-\frac{\alpha}{k}\right)}\right)}\right.\right.$$

$$\left.\left.+\alpha r\left(-e^{r\left(1-\frac{\alpha}{k}\right)}\right)+k e^{r\left(1-\frac{\alpha}{k}\right)}\right)+a\alpha k\right)/2k$$

Here are two examples of sets of parameters where the fixed point $(0, \alpha)$ is attracting and repelling.

Example 4.6: The moduli of both eigenvalues of $J_{(0,\alpha)}$ are less than 1 for a set of values $r = 2.5, a, k = \{-110.9, -98.4\}, \{-157.4, -38.9\}$, $\{-136.8, 80.6\}, \{200.6, -111.7\}, \{26, -72.7\}, \{-105.5, 218.5\}$, $\{-56.6, 72.9\}, \{-242.7, -18.7\}$,$\{213.9, -64.5\}, \{69.4, -175.7\}$ and its corresponding scaling factors (α, β) as $\{-2.1, -124.2\}, \{-188.7, -86.7\}$, $\{-2.1, -124.2\}, \{-188.7, -86.7\}$, {-167.7, 117.2}, $\{173.3,28.4\}$, $\{152.1,243.2\}, \{7.4, -120.5\}, \{21.9, -118.1\}, \{-176.7, 46\}$, $\{53.4, -9.3\}$, $\{9.9,142.0\}$ respectively.

Here, the fixed point $(0, \alpha)$ is locally asymptotically stable (attracting).

Example 4.7: The moduli of both eigenvalues of $J_{(0,\alpha)}$ are greater than 1 for a set of values $r = 5.5, a, k = \{-167.2, -209.6\}, \{-193.7,103\}$, $\{-179.8, -93.7\}, \{131.8,135\}, \{1.1, -148.8\}, \{-119.4, -38.4\}$, $\{-87.9, -158.2\}, \{-234.4,202\}$,$\{194.5, 45.1\}$, $\{42.3, -57.5\}$ and its corresponding scaling factors (α, β) as $\{142.6,191\}, \{204.2,198.5\}$,{97.1, 217.6}, $\{-120.8,30\}, \{-214.2, -75.1\}, \{-76.7,87.6\}, \{107.2, -32.4\}$, $\{45.8,241.1\}, \{26, -169.7\}, \{-77.4, -203.3\}$ respectively.

Here, the fixed point $(0, \alpha)$ is repelling.

4.4.4 Stability of fixed point (α, α)

While proving the existence of the fixed point (α, α), a computer simulation is done and found the space of parameters (a, r, k) with 5000 points as well as the scaling factors space (α, β) which is depicted in Fig. 4.12.

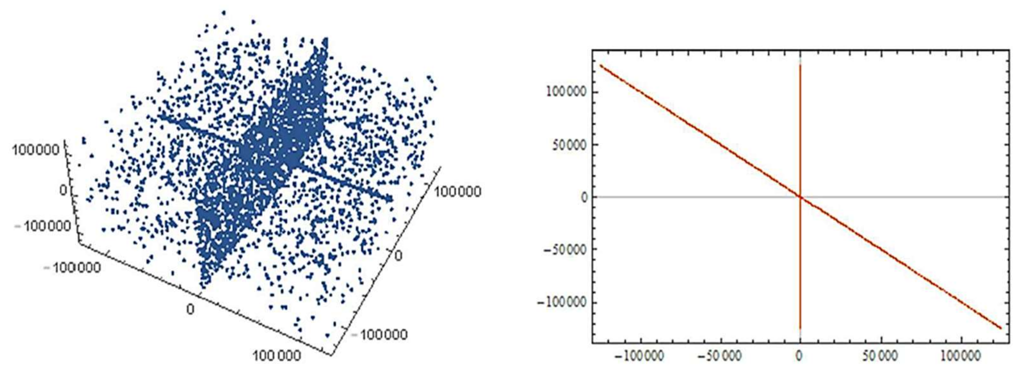


Fig.4.12. Plot of parameter space (a, r, k) and the scaling factor (α, β) of the scaled system Eqn. (4.2) where (α, α) is a fixed point.

The eigenvalues of the $J_{(\alpha,\alpha)}$ are

$$\left(e^{-a(\alpha+\beta)}\left(-\sqrt{\left(-2a\alpha k+2\alpha r e^{r\left(1-\frac{2\alpha}{k}\right)}-ke^{r\left(\left(1-\frac{2\alpha}{k}\right)\right)}\right)^{2}-4\left(2a\alpha k^{2}e^{a(\alpha+\beta)+r\left(1-\frac{2\alpha}{k}\right)}-4a\alpha^{2}kre^{r\left(1-\frac{2\alpha}{k}\right)}\right)}+2\alpha r\left(-e^{r\left(1-\frac{2\alpha}{k}\right)}\right)+ke^{r\left(1-\frac{2\alpha}{k}\right)}+2a\alpha k\right)/2k\right)$$

and

$$\left(e^{-a(\alpha+\beta)}\left(\sqrt{\left(-2a\alpha k+2\alpha r e^{r\left(1-\frac{2\alpha}{k}\right)}-ke^{r\left(\left(1-\frac{2\alpha}{k}\right)\right)}\right)^{2}-4\left(2a\alpha k^{2}e^{a(\alpha+\beta)+r\left(1-\frac{2\alpha}{k}\right)}-4a\alpha^{2}kre^{r\left(1-\frac{2\alpha}{k}\right)}\right)}+2\alpha r\left(-e^{r\left(1-\frac{2\alpha}{k}\right)}\right)+ke^{r\left(1-\frac{2\alpha}{k}\right)}+2a\alpha k\right)/2k\right)$$

Here are two examples of sets of parameters where the fixed point (α, α) is attracting and repelling.

Example 4.8: The moduli of both eigenvalues of $J_{(\alpha,\alpha)}$ are less than 1 for a set of values $r = 6.5, a, k = \{-55.9, 36.6\}, \{-236.7, -49\}$, $\{91.1, -7.2\}, \{74.6, -179.1\}, \{245.4, 28.7\}, \{110.5, 100.5\}, \{204, -116.7\}$,

$\{-155.1, -183.3\}, \{-247.9, -25.8\}, \{-105.5, 218.5\}$ and its corresponding scaling factors (α, β) as $\{-44.9, -32.4\}, \{-220, 38.6\}, \{183.5, 81.8\}$,

$\{-7.4, 84\}, \{86.3, -8.4\}, \{200.3, 212.3\}, \{152, -63.3\}, \{-75.3, -63\}$, $\{-1.1, -0.4\}, \{-120.5, -84.5\}$ respectively.

Here, the fixed point (α, α) is locally asymptotically stable (attracting).

Example 4.9: The moduli of both eigenvalues of $J_{(\alpha,\alpha)}$ are greater than 1 for a set of values $r = 6.5, a, k = \{-167.2, -209.6\}, \{-193.7, 103\}$, $\{-179.8, -93.7\}, \{131.8, 135\}, \{1.1, -148.8\}, \{-119.4, -38.4\}$, $\{-87.9, -158.2\}, \{-234.4, 202\}, \{194.5, 45.1\}$, $\{42.3, -57.5\}$ and its corresponding scaling factors (α, β) as $\{142.6, 191\}, \{204.2, 198.5\}$, $\{97.1, 217.6\}, \{-120.8, 30\}, \{-214.2, -75.1\}, \{-76.7, 87.6\}, \{107.2, -32.4\}$, $\{45.8, 241.1\}, \{26, -169.7\}, \{-77.4, -203.3\}$ respectively.

Here, the fixed point (α, α) is repelling.

Conjecture 4. 1: If $\beta = -\alpha$, then (α, α) is a fixed point of the scaled Nicholson-Bailey system Eqn. (4.2) where α and β are arbitrary scaling factors.

4.4.5 Stability of fixed point $(\boldsymbol{\alpha}, \boldsymbol{\beta})$

In establishing the existence of the fixed point (α, β), a computer simulation is done and found the space of parameters (a, r, k) with 5000 points as well as the scaling factors space (α, β) which is depeicted in Fig. 4.13.

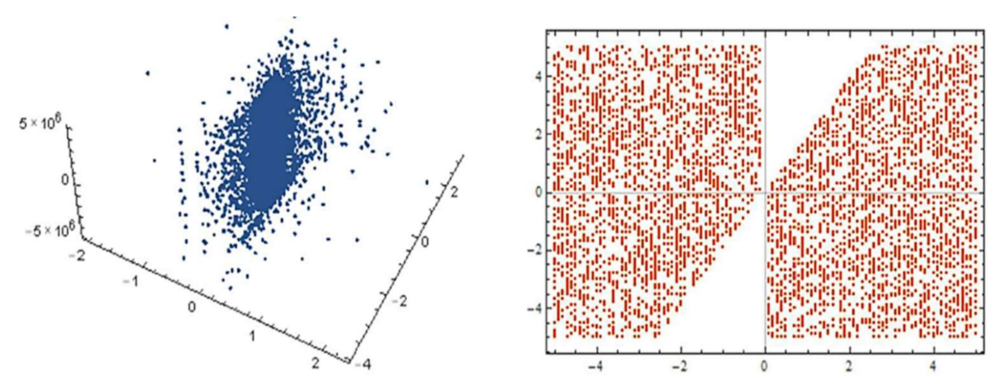


Fig.4.13. Plot of parameter space $(\boldsymbol{a}, \boldsymbol{r}, \boldsymbol{k})$ and the scaling factor $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ of the scaled system Eqn. (4.2) where $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is a fixed point.

The eigenvalues of $J_{(\alpha,\beta)}$ are

$$\left(e^{-2a\beta)}\left(-\sqrt{\left(-2a\alpha k+2\alpha r e^{r\left(1-\frac{2\alpha}{k}\right)}-ke^{r\left(\left(1-\frac{2\alpha}{k}\right)\right)}\right)^{2}-4\left(2a\alpha k^{2}e^{2a\beta+r\left(1-\frac{2\alpha}{k}\right)}-4a\alpha^{2}kre^{r\left(1-\frac{2\alpha}{k}\right)}\right)}\right.\right.$$
$$\left.\left.+2\alpha r\left(-e^{r\left(1-\frac{2\alpha}{k}\right)}\right)+ke^{r\left(1-\frac{2\alpha}{k}\right)}+2a\alpha k\right)/2k\right)$$

and

$$\left(e^{-2a\beta)}\left(\sqrt{\left(-2a\alpha k+2\alpha r e^{r\left(1-\frac{2\alpha}{k}\right)}-ke^{r\left(\left(1-\frac{2\alpha}{k}\right)\right)}\right)^{2}-4\left(2a\alpha k^{2}e^{2a\beta+r\left(1-\frac{2\alpha}{k}\right)}-4a\alpha^{2}kre^{r\left(1-\frac{2\alpha}{k}\right)}\right)}\right.\right.$$
$$\left.\left.+2\alpha r\left(-e^{r\left(1-\frac{2\alpha}{k}\right)}\right)+ke^{r\left(1-\frac{2\alpha}{k}\right)}+2a\alpha k\right)/2k\right)$$

Here are two examples of sets of parameters where the fixed point (α, β) is attracting and repelling.

Example 4.10: The moduli of both the eigenvalues of $J_{(\alpha,\beta)}$ are less than 1 for a set of values $r = 6.5, a, k = \{-105.5,218.5\}, \{-157.4, -38.9\}$, $\{-110.9, -98.4\}, \{77.4,188\}, \{-56.6,72.9\}, \{-55.9,36.6\}, \{-56.6,72.9\}$, $\{-55.9,36.6\}, \{150.4,65.5\}, \{69.4, -175.7\}$, $\{-247.9, -25.8\}, \{204, -116.7\}$ and its corresponding scaling factors (α, β) as $\{7.4, -120.5\}$, $\{-188.7, -86.7\}, \{-2.1, -124.2\}, \{-116,181.9\}, \{21.9, -118.1\}$, $\{-214.1, -44.9\}, \{-77.9,117.7\}, \{9.9,142\}, \{236.9, -1.1\}, \{-120.1,152\}$ respectively.

Here, the fixed point (α, β) is locally asymptotically stable (attracting).

Example 4.11: The moduli of both eigenvalues of the Jacobian $J_{(\alpha,\beta)}$ are greater than 1 for a set of values $r = 6.5\ a, k =$ $\{-167.2, -209.6\}, \{-193.7,103\}, \{-179.8, -93.7\}, \{131.8,135\}$,

$\{1.1, -148.8\}, \{-119.4, -38.4\}, \{-87.9, -158.2\}, \{-234.4, 202\}$, $\{194.5, 45.1\}$, $\{42.3, -57.5\}$ and its corresponding scaling factors (α, β) as $\{142.6, 191\}, \{204.2, 198.5\}, \{97.1, 217.6\}, \{-120.8, 30\}, \{-214.2, -75.1\}$, $\{-76.7, 87.6\}, \{107.2, -32.4\}, \{45.8, 241.1\}, \{26, -169.7\}, \{-77.4, -203.3\}$ respectively. Here, the fixed point (α, β) is repelling.

From Fig.4.13, the following conjecture can be claimed.

Conjecture 4.2: The fixed point (α, β) of the scaled Nicholson-Bailey system Eqn. (4.2) where α and β are arbitrary scaling factors is symmetric about the line $\beta = \alpha$ in the $\alpha\beta$ plane.

In a similar way, the local asymptotic stability of the other fixed points can be comprehended.

4.5 Periodic Solutions

A solution $\{x_n, y_n\}_n$ of a two dimensional dynamical system is said to be periodic of period t if (x_{n+t}, y_{n+t}) =(x_n, y_n) for any given initial conditions. A solution $\{x_n, y_n\}_n$ is periodic with a prime period p , provided p is the smallest positive integer having this property [29].

A solution $\{x_n, y_n\}_n$ of a two dimensional dynamical system is said to be quasi-periodic of period t if $(x_{n+t}, y_{n+t}) \cong (x_n, y_n)$ for the given initial conditions. In this study, computer simulation was conducted and obtained a set of periodic solution for system Eqn. (4.1). Further, compared the same with system Eqn. (4.2). Figures Fig. 4.14.1-Fig. 4.14.5 and Tables 4.2 depicts the respective results.

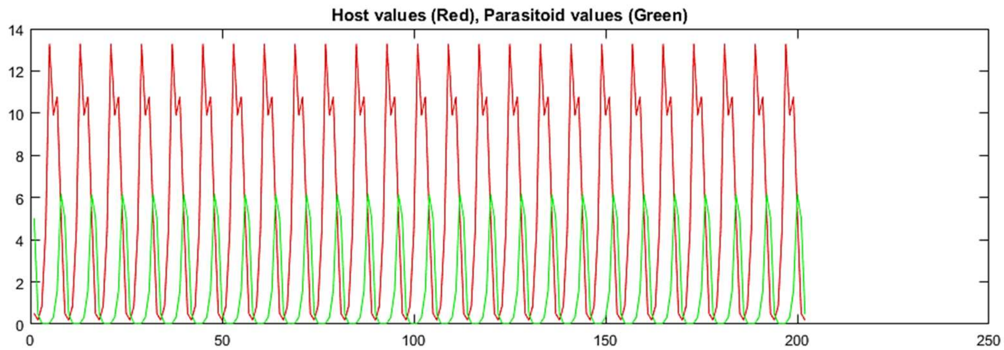


Fig.4.14.1. Periodic solutions of Nicholson – Bailey model Eqn. (4.1) for $(a, k, r) = (0.54, 11.55, 1.8)$ and initial values $(x, y) = (18, 2)$ with period 8.

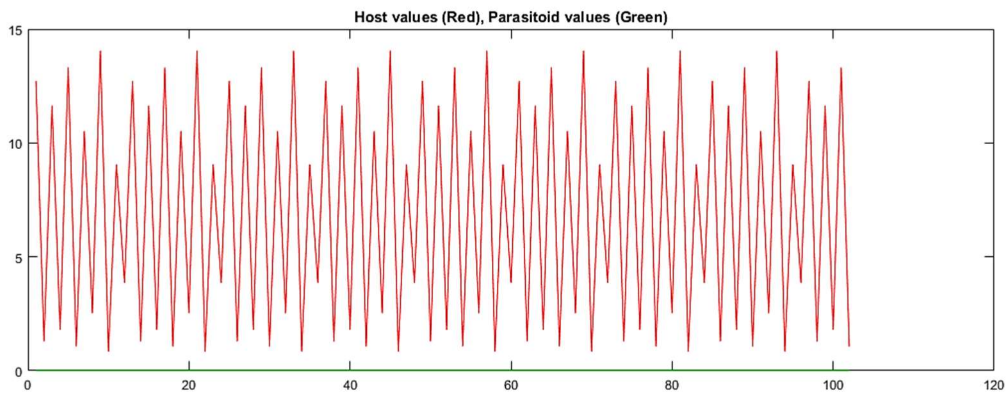


Fig.4.14.2. Periodic solutions of Nicholson – Bailey model Eqn. (4.1) for $(a, k, r) = (32.72, 21.14, 2.69)$ and initial values $(x, y) = (7.8, 3.2)$ with period 16.

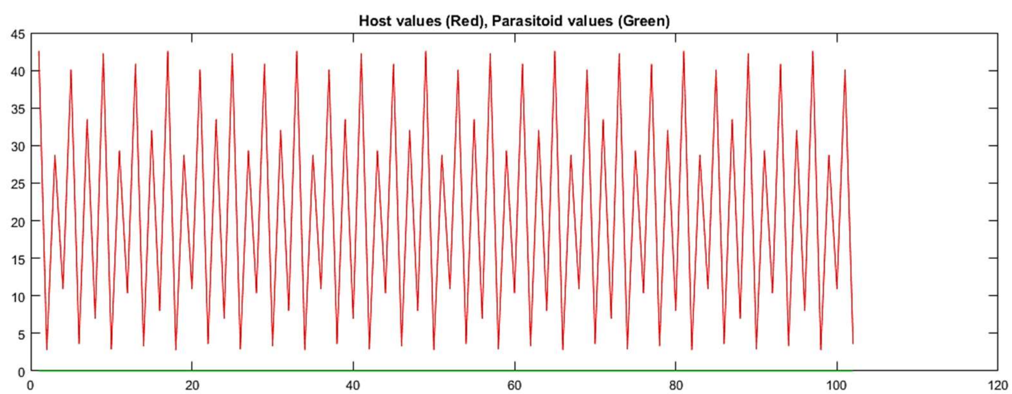


Fig.4.14.3. Periodic solutions of Nicholson – Bailey model Eqn. (4.1) for $(a, k, r) = (0.44, 6.89, 2.71)$ and initial values $(x, y) = (10.5, 46)$ with period 12.

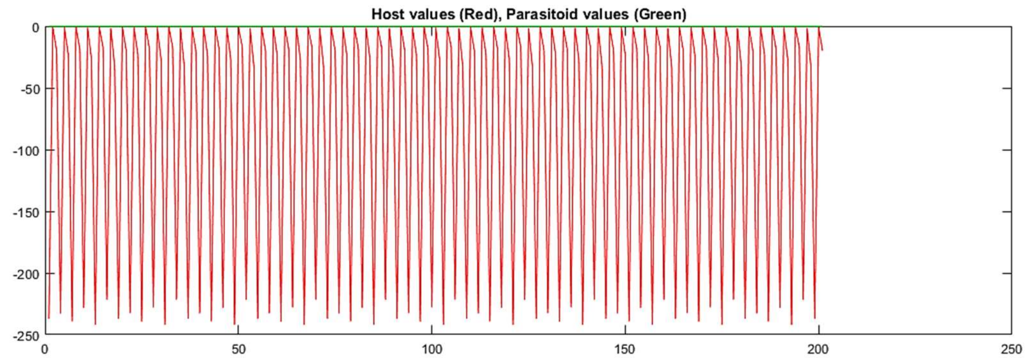


Fig.4.14.4. Periodic solutions of Nicholson – Bailey model Eqn. (4.1) for $(a, k, r) = (17.75, -85.81, 3.2)$ and initial values $(x, y) = (-4, 27)$ with period 35.

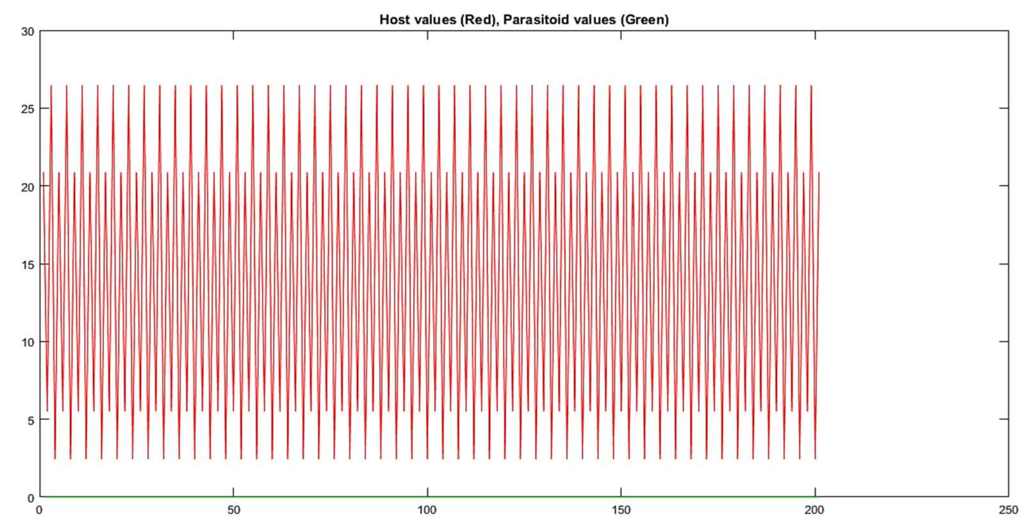


Fig.4.14.5. Periodic solutions of Nicholson – Bailey model Eqn. (4.1) for $(a, k, r) = (-2.82, 13.82, 2.61)$ and initial values $(x, y) = (7, -37)$ with period 71.

A set of periodic trajectories (with high periods) of system Eqn. (4.1) were found. Corresponding to these parameters and initial values, the study investigated the scaled system Eqn. (4.2) and found a few set of scaling factors α and β for which periodic solutions are achieved. In most of the cases, with different sets of parameters and different scaling factors, either trajectory converges or diverges. It is noted that there are trajectories which either converge or diverge even though the scaling factors are chosen very close to zero are tabulated in Table 4.2.

Table 4.2: Solutions of scaled Nicholson-Bailey model Eqn. (4.2) corresponding to parameters and initial values as stated in Fig. 4.14.1-Fig. 4.14.5 for the Nicholson-Bailey model Eqn. (4.1).

Case	Scaling factor	Scaling factor	Scaling factor	Scaling factor
$r = 1.8, a = 0.54$ $k = 11.55$ *and* $x = 18, y = 2$	$\alpha = 97, \beta = -100$ Remark: Trajectory diverges	$\alpha = 59, \beta = 57$ Remark: Converges to $(0, 59)$	$\alpha = 1, \beta = 1$ Remark: Converges to $(1.8028, 2.3416)$	$\alpha = 59, \beta = 57$ Remark: Quasi-periodic of period 5
$r = 2.69, a = 32.72$ $k = 21.14$ *and* $x = 7.8, y = 3.2$	$\alpha = 42, \beta = -6$ Remark: Trajectory diverges	$\alpha = 5.25, \beta = 9.49$ Remark: Converges to $(5.25, 0)$	$\alpha = 0.001, \beta = 0.005$ Remark: Periodic of period 6	$\alpha = 0.01, \beta = 0.005$ Remark: Periodic of period 5
$r = 2.71, a = 0.44$ $k = 6.89$ *and* $x = 10.5, y = 46$	$\alpha = 0.01, \beta = 0.05$ Remark: Converges to $(3.8268, 2.6852)$	$\alpha = 66, \beta = 17$ Remark: Converges to $(0, 66)$	$\alpha = -43, \beta = 52$ Remark: Trajectory diverges	$\alpha = 0.00003, \beta = 0.00001$ Remark: Converges to $(3.8784, 2.6921)$
$r = 3.2, a = 17.75$ $k = -85.81$ *and* $x = -4, y = 27$	$\alpha = -24, \beta = 53$ Remark: Converges to $(0, -24)$	$\alpha = 9, \beta = -73$ Remark: Trajectory diverges	$\alpha = 0, \beta = 5 \times 10^{-9}$ Remark: Trajectory diverges	$\alpha = 0.1067, \beta = 0.9619$ Remark: Trajectory diverges
$r = 2.61, a = -2.82$ $k = 13.82$ *and* $x = 7, y = -37$	$\alpha = -85, \beta = -52$ Remark: Converges to $(0, -85)$	$\alpha = -81, \beta = -26$ Remark: Trajectory diverges	$\alpha = 5 \times 10^{-11}, \beta = -81$ Remark: Converges to $(0, 5 \times 10^{-11})$	$\alpha = 5 \times 10^{-11}, \beta = 5 \times 10^{-11}$ Remark: Trajectory diverges

4.6 Bifurcation Analysis

The behavior of the dynamical system changes drastically with the change of system parameters. The changes are not only quantitative such as the changes in the location of a fixed points, but also qualitative. Points can be created or destroyed, there may be change in their stability, and the system behavior can change from stationary or periodic to chaotic. It is the qualitative changes in the system dynamics that are quite pertinent for investigation in dynamical systems[40].

Here, performed bifurcation analysis on the Nicholson-Bailey models Eqn. (4.1) and the scaled Nicholson-Bailey model Eqn. (4.2).

Fixing $a = 0.2, k = 22.47, and\ r$ as the parameter for the bifurcation analysis, which is varying over the interval $[-10,10]$, the following bifurcation plot in Fig. 4.15 is plotted by displaying the last 250 points of the simulated set of 1000 points for 200 distinct r values over the specified interval.

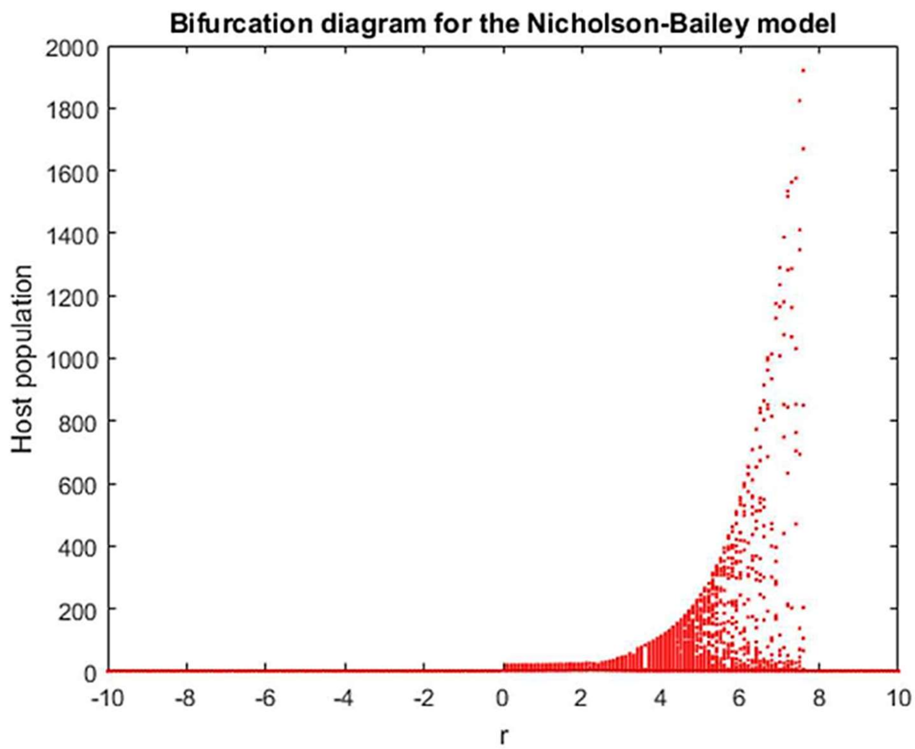


Fig.4. 15. Bifurcation diagram of the parameter r for the Nicholson-Bailey model.

It is seen that in the Eqn. (4.1), when the parameter *r* lies in the interval (2,6.75), then the fixed points became unstable having different kinds of behaviors (periodic/chaotic) [41]. In the scaled system Eqn. (4. 2), for the scaling factors $(\alpha, \beta) = (0,20), (1,0), (1,1), (-1,-1),$ $(-1,-2), (1,20), (5,20), (12,20), and\ (15,20)$, the fixed points are unstable whenever the parameter *r* belongs to the interval (5,12), (4,10), (3,10), (2,4) etc.

It shows that the parameter *r* is quite sensitive to the scaling factors which is showing in the following Fig. 4.16.1-Fig. 4.16.3. None of these diagrams is similar to others.

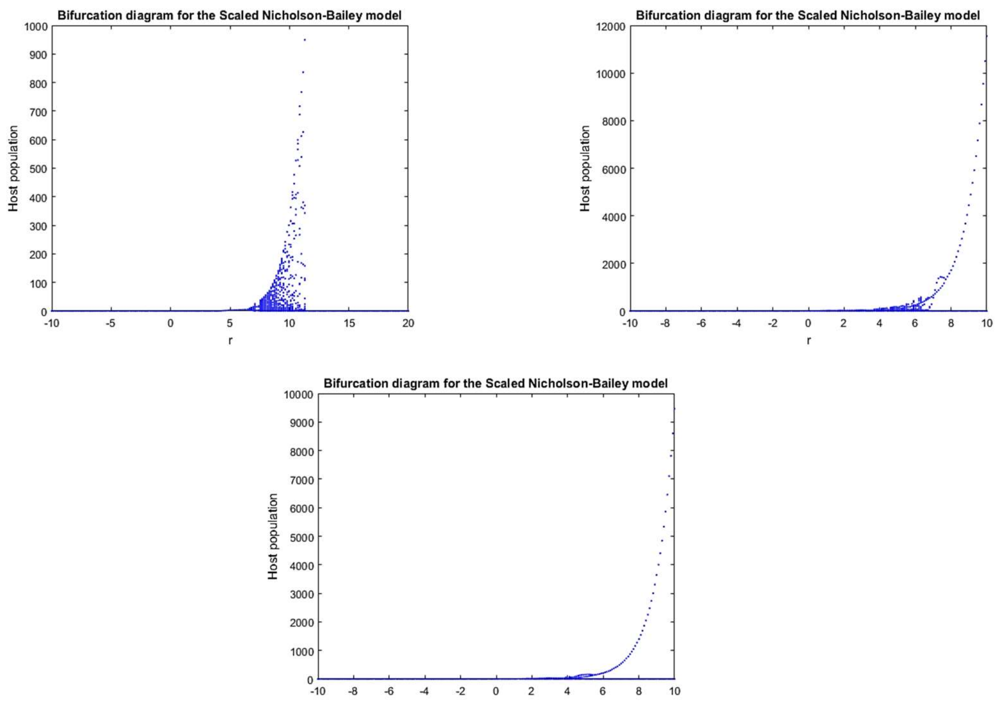


Fig. 4. 16.1 Bifurcation diagram for different scaling factors $(\alpha, \beta) = (0, 20), (1, 0), (1, 1)$ respectively for the scaled Nicholson Bailey model.

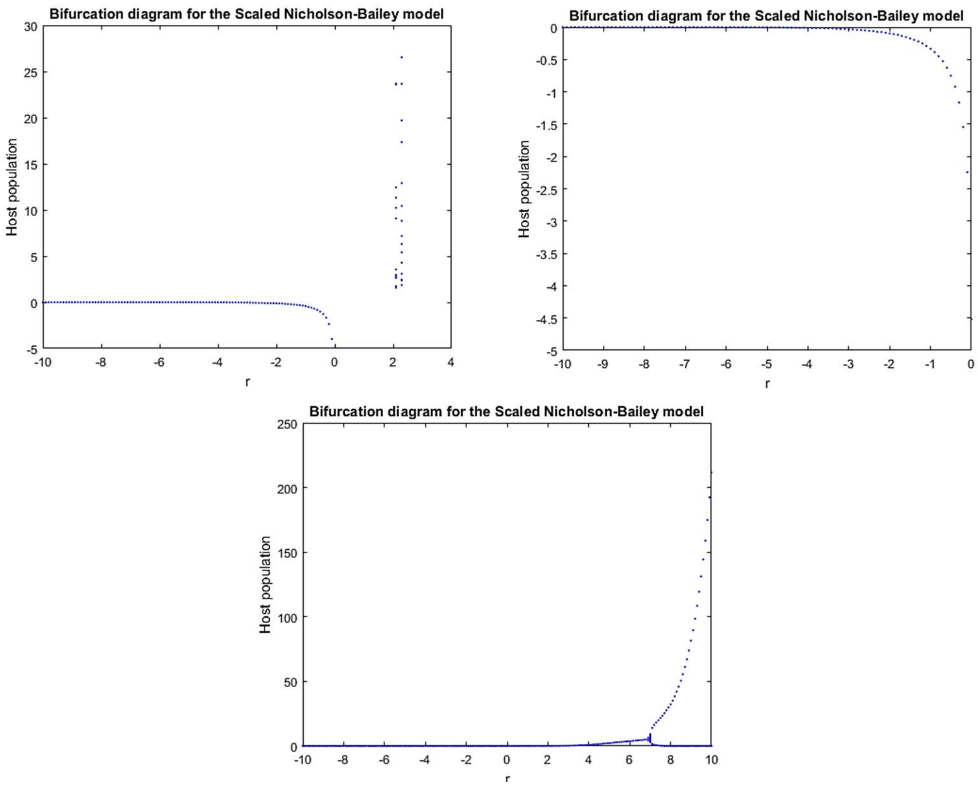


Fig. 4. 16.2 Bifurcation diagram for different scaling factors $(\alpha, \beta) = (-1, -1), (-1, -2), (1, 20)$ respectively for the scaled Nicholson Bailey model.

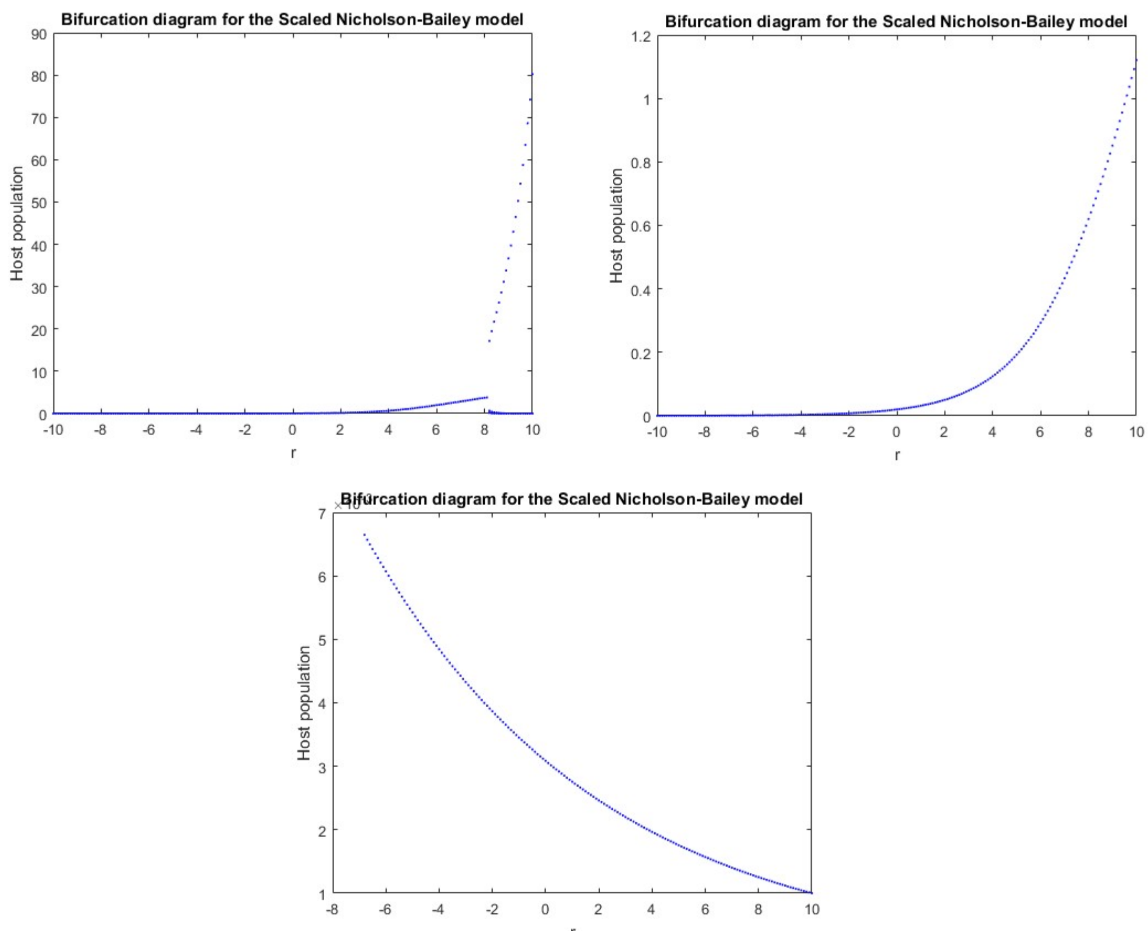


Fig. 4. 16.3 Bifurcation diagram for different scaling factors $(\alpha, \beta) = (5, 20), (12, 20), (15, 20)$ respectively for the scaled Nicholson Bailey model.

4.7 Chaotic Solutions

The work explores the existence of chaotic solution of the dynamical system Eqn. (4.1) and its scaled model Eqn. (4.2). Computationally, first some chaotic solutions of the Nicholson-Bailey model Eqn. (4.1) were obtained and then explored the trajectory behavior of the scaled model Eqn. (4.2) corresponding to the parameters specified in each of the cases as shown in the Figures Fig. 4.17.1 – Fig. 4.17.5.

In doing so, in each case, we keep all the parameters fixed including the initial values to see what happens to the trajectory behavior while the scaling factors

α and β are changed. The detail results are adumbrated in Figures Fig. 4.18.1 – Fig. 4.18.4. The largest Lyapunov exponent is calculated for all such situations of the dynamical system Eqn. (4.1) numerically to ensure the trajectories are indeed chaotic [102].

It is detected that the sensitivity of chaotic behavior of trajectories does not depend on the initial values, but rather depends on the scaling factors. It is seen in the scaled model that the trajectories are behaving differently (converging, diverging) for those set of parameters for which the Nicholson-Bailey model Eqn. (4.1) possessed chaotic solutions.

The chaotic behavior of trajectories is ensured by computing the largest Lyapunov exponent which is shown in Fig. 4.17.1 – Fig 4.17.5 and Fig. 4.18.1-Fig. 4.18.4.

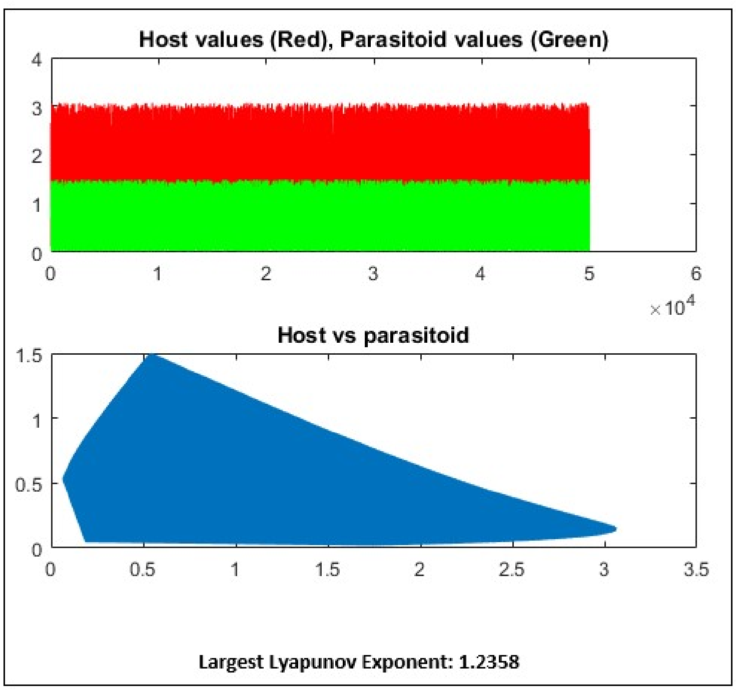


Fig. 4. 17.1 The trajectory of chaotic solution for the Nicholson-Bailey model Eqn (4.1) for the parameter values $(r, a, k) = (2.6, 2.7, 2)$ and initial values $x = 1.48, y = 0.7$.

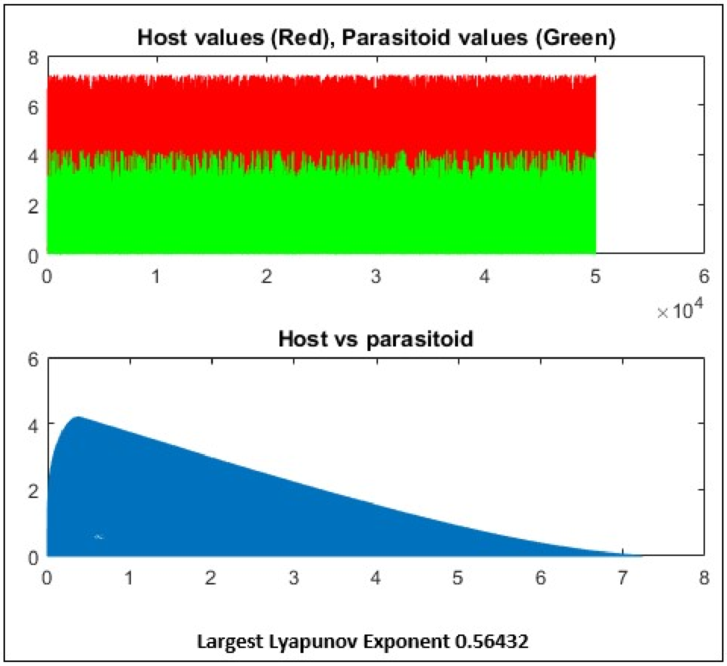


Fig. 4. 17.2 The trajectory of chaotic solution for the Nicholson-Bailey model Eqn (4.1) for the parameter values $(r, a, k) = (2.6, 3.4, 3.8)$ and initial values $x = 0.3, y = 0.5$.

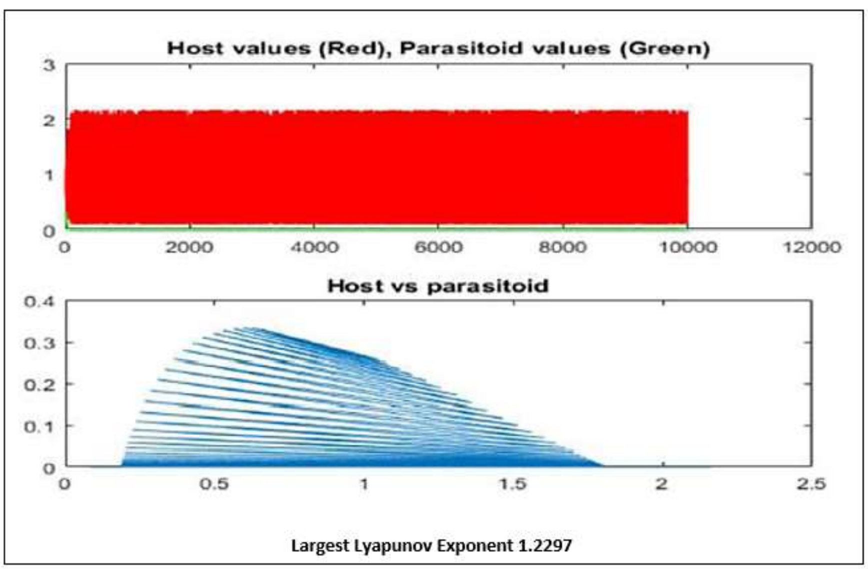


Fig. 4. 17.3 The trajectory of chaotic solution for the Nicholson-Bailey model Eqn (4.1) for the parameter values $(r, a, k) = (2.8, 1.5, 1)$ and initial values $x = 0.6948, y = 0.3171$.

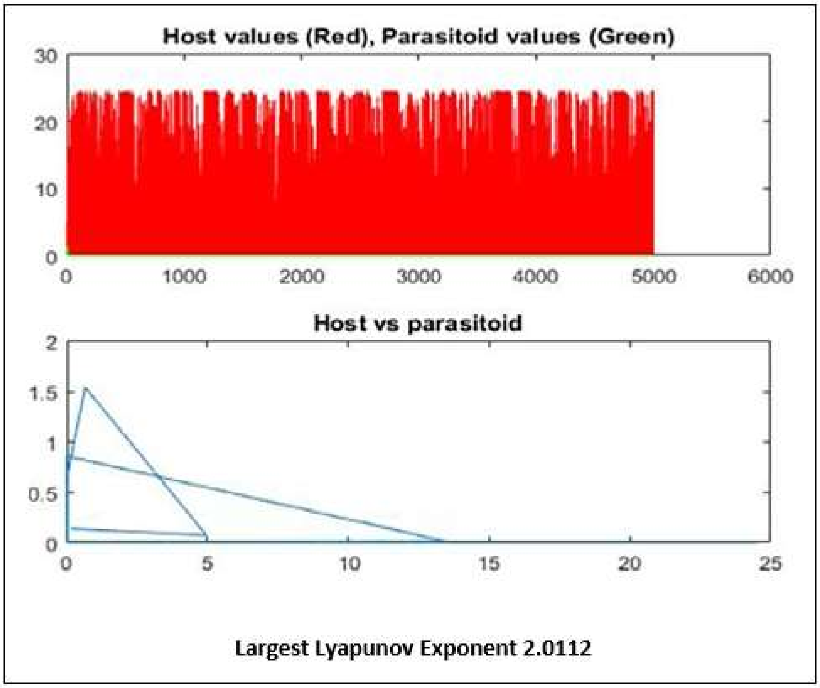


Fig. 4. 17.4 The trajectory of chaotic solution for the Nicholson-Bailey model Eqn (4.1) for the parameter values $(r, a, k) = (4.4, 5.1, 3.6)$ and initial values $x = 0.1455, y = 0.1361$.

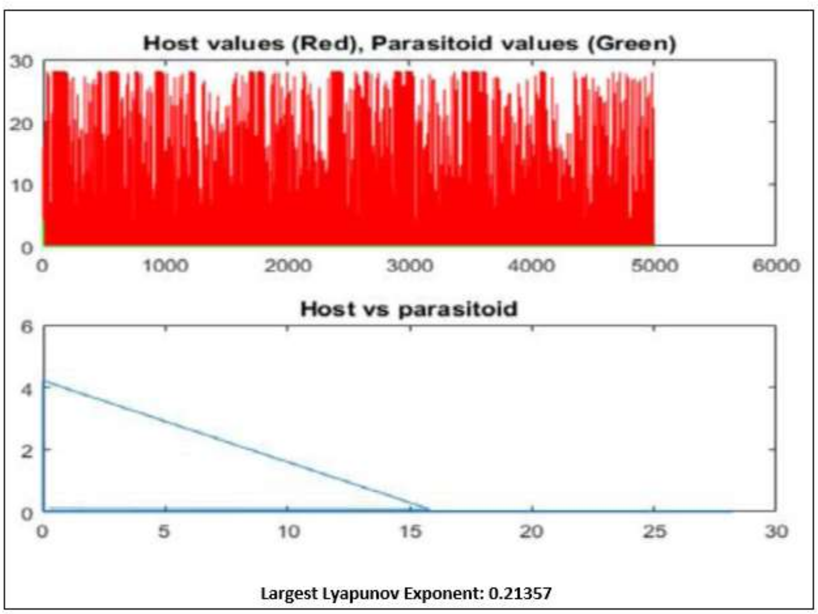


Fig. 4. 17.5 The trajectory of chaotic solution for the Nicholson-Bailey model Eqn (4.1) for the parameter values $(r, a, k) = (5.2, 3.6, 2.2)$ and initial values $x = 0.2399, y = 0.1233$.

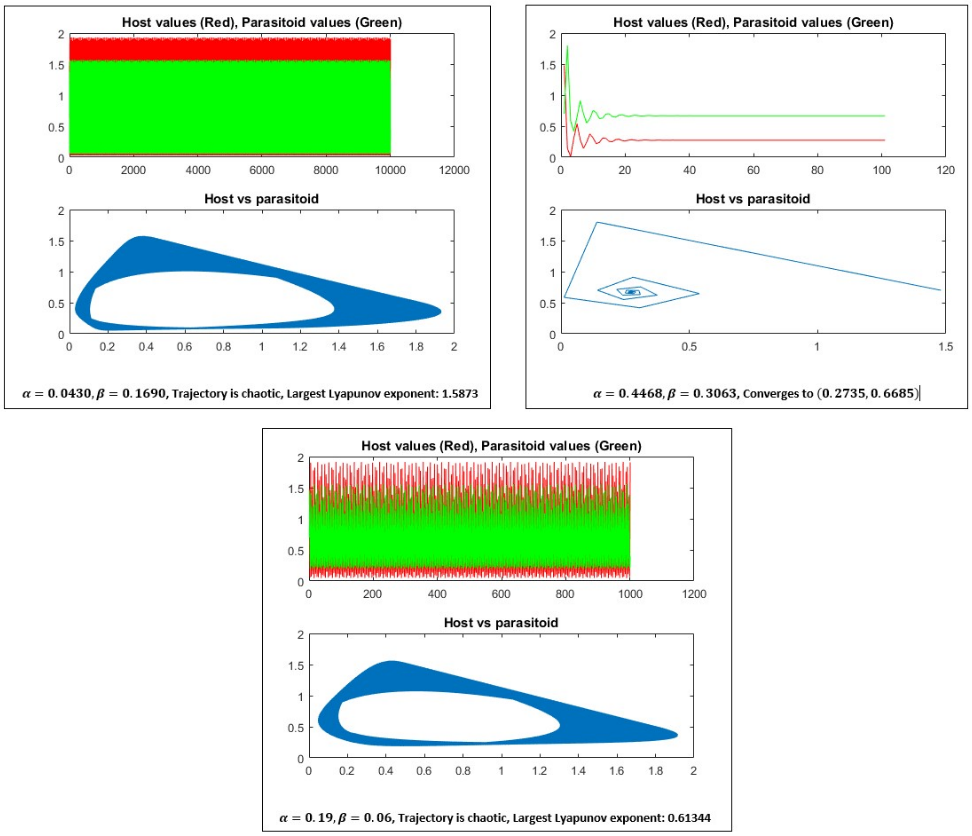


Fig. 4.18.1. Solutions of the scaled Nicholson-Bailey model Eqn. (4.2) corresponding to $(r, a, k, x, y) = (2.6, 2.7, 2, 1.48, 0.7)$ of the Nicholson-Bailey model Eqn. (4.1).

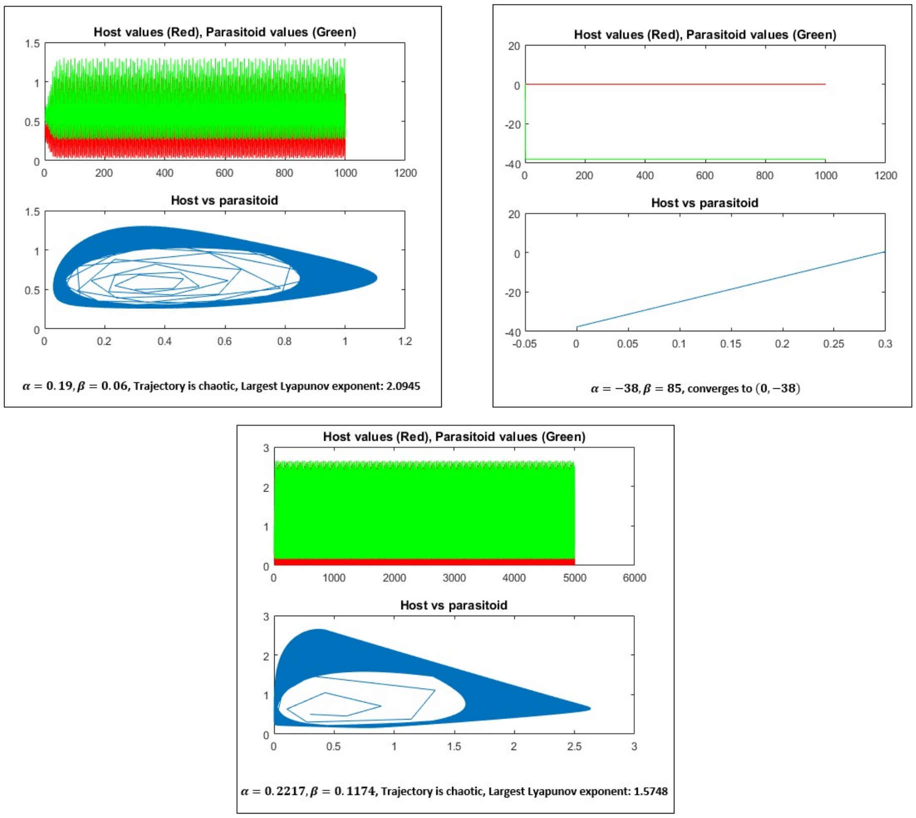
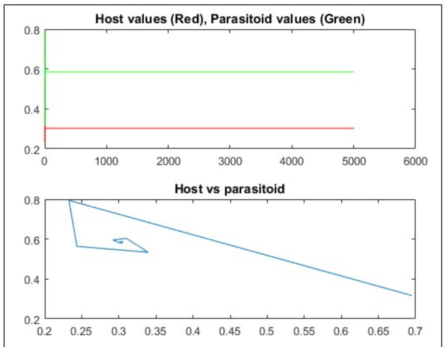
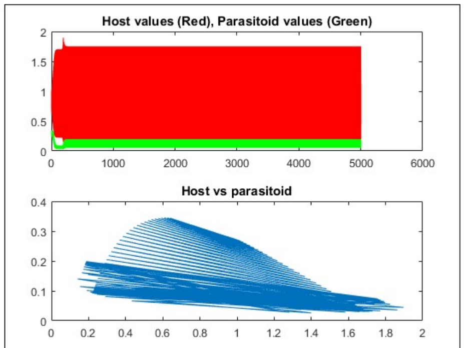


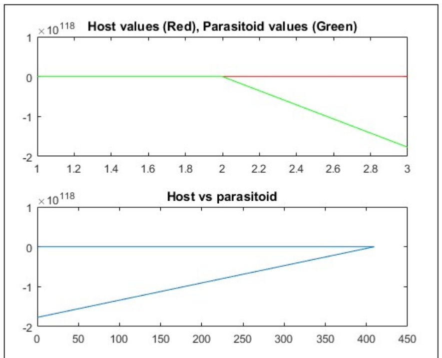
Fig. 4.18.2. Solutions of the scaled Nicholson-Bailey model Eqn. (4.2) corresponding to $(r, a, k, x, y) = (2.6, 3.4, 3.8, 0.3, 0.5)$ of the Nicholson-Bailey model Eqn. (4.1).



$\alpha = 0.4242, \beta = 0.5079$, Trajectory converges to $(0.3027, 0.5860)$



$\alpha = 0.005, \beta = 0.006$, Trajectory is chaotic, Largest Lyapunov exponent: 0.76523.



$\alpha = 0, \beta = -4$, Trajectory diverges

Fig. 4.18.3. Solutions of the scaled Nicholson-Bailey model Eqn. (4.2) corresponding to $(r, a, k, x, y) = (2.8, 1.5, 1, 0.6948, 0.3171)$ of the Nicholson-Bailey model Eqn. (4.1).

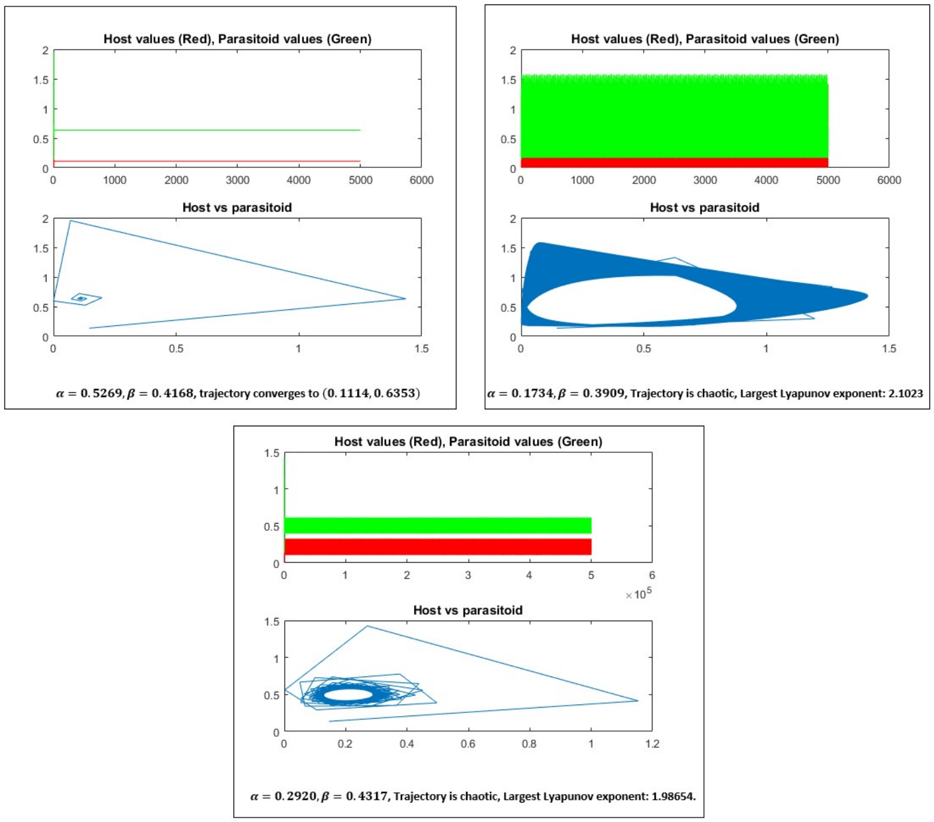


Fig. 4.18.4. Solutions of the scaled Nicholson-Bailey model Eqn. (4.2) corresponding to $(r, a, k, x, y) = (4.4, 5.1, 3.6, 0.1455, 0.1361)$ of the Nicholson-Bailey model Eqn. (4.1).

4.8 Nicholson-Bailey Model with Uniformly Distributed Noise

None of the models is perfect. In fact, most of the models are noisy. A Nicholson-Bailey model is created with with noise, taken as a random number in (0,1) interval following a uniform distribution. Therefore the Nicholson-Bailey model with uniform noise $(v_1, v_2) \in (0,1)$ X $(0,1)$ becomes the following system:

$$\begin{cases} x_{n+1} = x_n \left(e^{r\left(1-\frac{x_n}{k}\right)-ay_n} \right) + v_1 \\ y_{n+1} = x_n(1 - e^{-y_n}) + v_2 \end{cases} \qquad \text{Eqn. (4.14)}$$

4.8.1 Local stability of fixed points

The points (0,0) and $(k, 0)$ are no longer fixed points of this system Eqn. (4.14). It is trivial to observe that the fixed points (0,0) and $(k, 0)$ can be obtained only when there is no noise.

Theorem 4.6. The dynamical system Eqn. (4.14) possesses a fixed point $(0, v_2)$ if and only if $v_1 = 0$.

Proof. It is trivial if $(0, v_2)$ is a fixed point of the system Eqn. (4.14) then when $x = 0$ and $y = v_2$ we have $0 = v_1$ and $v_2 = v_2$. Eqn. (4.15)
Hence the point $(0, v_2)$ is a fixed point if and only if $v_1 = 0$ follows from the Eqn. (4.15).

In a similar manner, it can be proved that $(0, v_1)$ is a fixed point to the system Eqn.(4.14) if $v_1 = v_2$.

Clearly, the fixed points of Nicholson-Bailey system with noise is effected by the noise (v_1, v_2). Here a few examples of the fixed points were listed which are functions of the noise (v_1, v_2) including the corresponding parameters.

The following Table 4.3 shows that there are fixed points which are functions of the uniformly distributed noise (v_1, v_2) as well as there are fixed points which are not explicitly functionally related to the noise (v_1, v_2).

Table 4.3. Set of parameters a, k, r with uniformly distributed noise (v_1, v_2) of the noisy model Eqn. (4.14) for which different fixed points are achieved.

Parameters a, κ, r	Uniformly distributed noise	Fixed point	Remark
$a = 44$, $\kappa = 45$ and $r = -14$	$(\nu_1, \nu_2) = (0.6463, 0.7094)$	$(0.6463, 1.3557)$	Modulus of the eigenvalues about the fixed point are very less than 1 and so the fixed point is attracting.
$a = -0.124415$, $\kappa = -3751$ and $r = -0.131186$	$(\nu_1, \nu_2) = (0.5, 0.7)$	$(-4.5, 1.9)$	Moduli of both eigenvalues about the fixed point are greater than 1 and so the fixed point is repelling.
$a = 2.3105$, $\kappa = 4958$ and $r = 1.1785$	$(\nu_1, \nu_2) = (0.5, 0.7)$	$(0.6039, 1.2719)$	Moduli of both eigenvalues about the fixed point are less than 1 and so the fixed point is attracting.
$a = -0.5628$, $\kappa = -4119$ and $r = 0.7404$	$(\nu_1, \nu_2) = (0.5, 0.7)$	$(-4.3780, -5.2498)$	Moduli of both eigenvalues about the fixed point are less than 1 and so the fixed point is attracting.
$a = -0.7$, $\kappa = 65.4$ and $r = -8.2$	$(\nu_1, \nu_2) = (0.5, 0.3)$	$(0.3, 0.5)$	Moduli of both eigenvalues about the fixed point are less than 1 and so the fixed point is attracting.

4.8.2 Computational Bifurcation Analysis

Here the local bifurcation analysis of the fixed points of Nicholson-Bailey model with uniformly distributed noise Eqn. (4.14) was presented.

Fixing $a = 0.2, k = 22.47$ and r, as the parameter of the bifurcation analysis, which is varying over the interval $[-10, 10]$, a bifurcation plot is plotted by displaying the last 250 points of the simulated set of 1000 points for 200 distinct r values over the specified interval.

The qualitative behavior of the dynamics of the noisy model Eqn. (4.14) gets changed due to one of the control parameters r drastically, as seen in the bifurcation diagrams in Fig. 4.19. For a different choice of noises, the fixed points are becoming unstable (either periodic/chaotic) as r changes over the interval (4,10) whereas in the original system Eqn. (4.1) the instability of the fixed points was restricted to the interval (2,6.75).

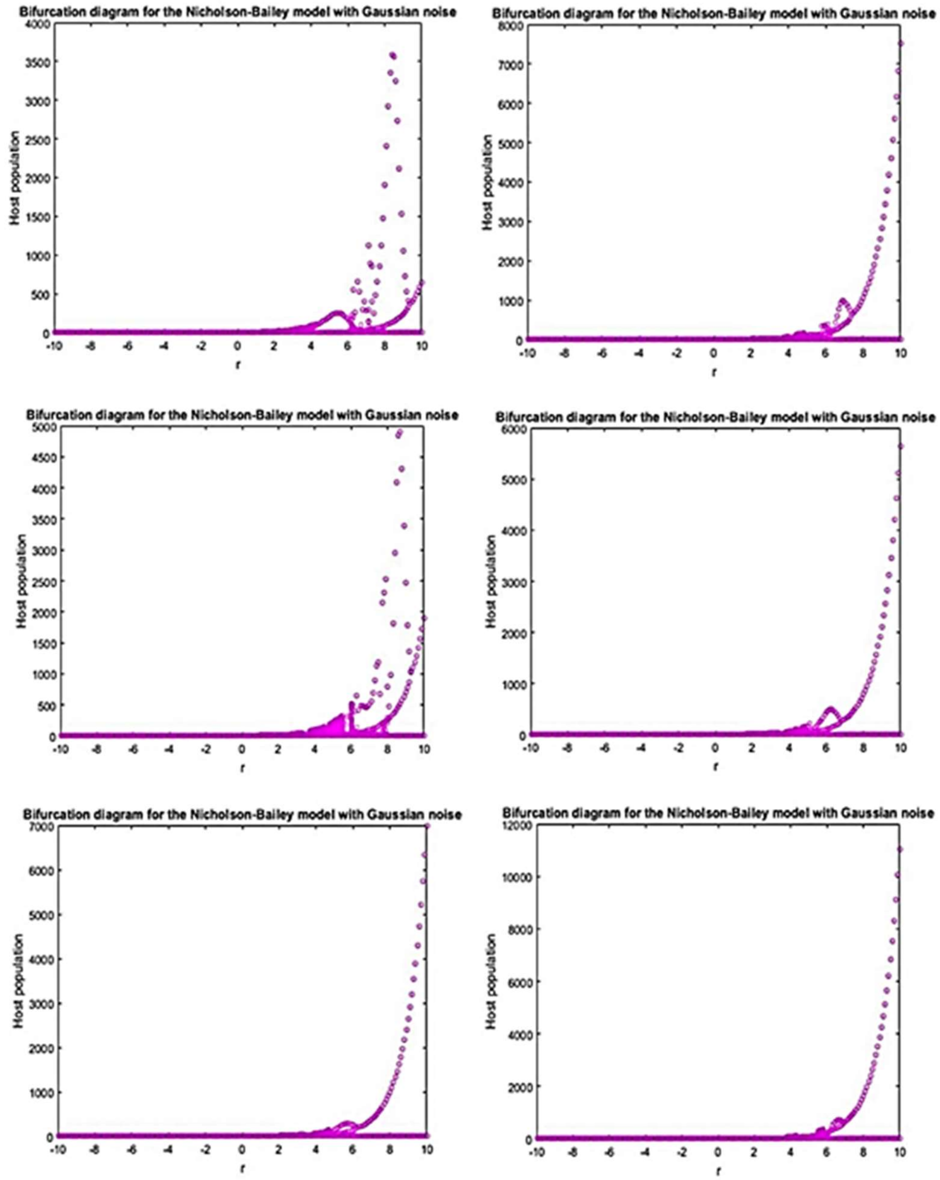


Fig. 4. 19. Bifurcation diagram for different noises for the Nicholson-Bailey model with uniformly distributed noise over $(0, 1)$.

4.8.3 Periodic Solutions

In this section, the investigation into the existence of any periodic solution of the model with uniformly distributed noise Eqn. (4.14) was presented. First, we shall compare the solution of the noisy model Eqn. (4.14) with those set of parameters for which the Nicholson-Bailey model possesses periodic solutions as stared in Fig. 4.14.1-4.14.5. Further, computationally discovered a few set of periodic solutions of the system Eqn. (4.14), which are shown in the figures Fig. 4.20.1 – Fig. 4.20.4. There are examples of parameters as shown in Fig. 4.19.1 – Fig. 4.19.4 for which the noisy model does not have any periodic solution whereas the original Nicholson-Bailey model possessed periodic solution. In such cases, the solution of the noisy model Eqn. (4.14) is either convergent or divergent. One set of parameters was obtained for which the Nicholson-Bailey model has periodic solution, as shown in Fig. 4.19.1 whereas model Eqn. (4.14) has periodic solutions for three different set of noises. Also the study searches for parameters with different noises for which the system Eqn. (4.14) possessed periodic solutions of different periods *viz.* $3, 4$ and 6 as depicted in the figures Fig. 4.20.1 – Fig. 4.20.4.

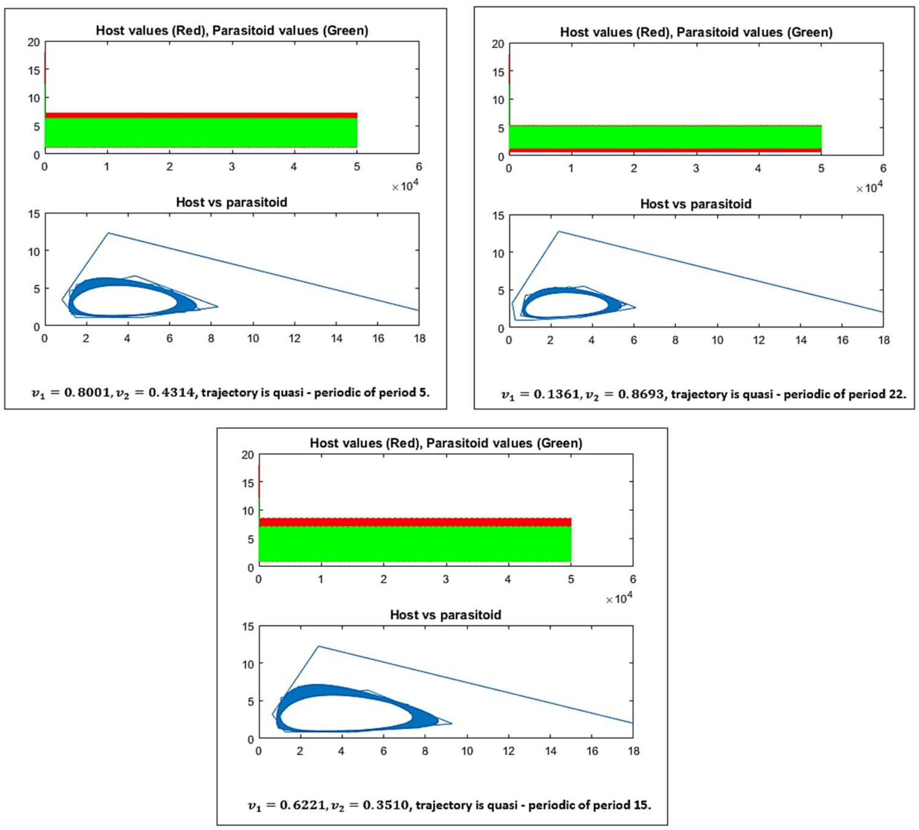


Fig. 4.19.1. Solution of Nicholson-Bailey model with uniformly distributed noise Eqn. (4.14) corresponding to the $(r, a, k, x, y) = (1.8, 0.54, 11.55, 18, 2)$ of the Nicholson-Bailey model Eqn. (4.1).

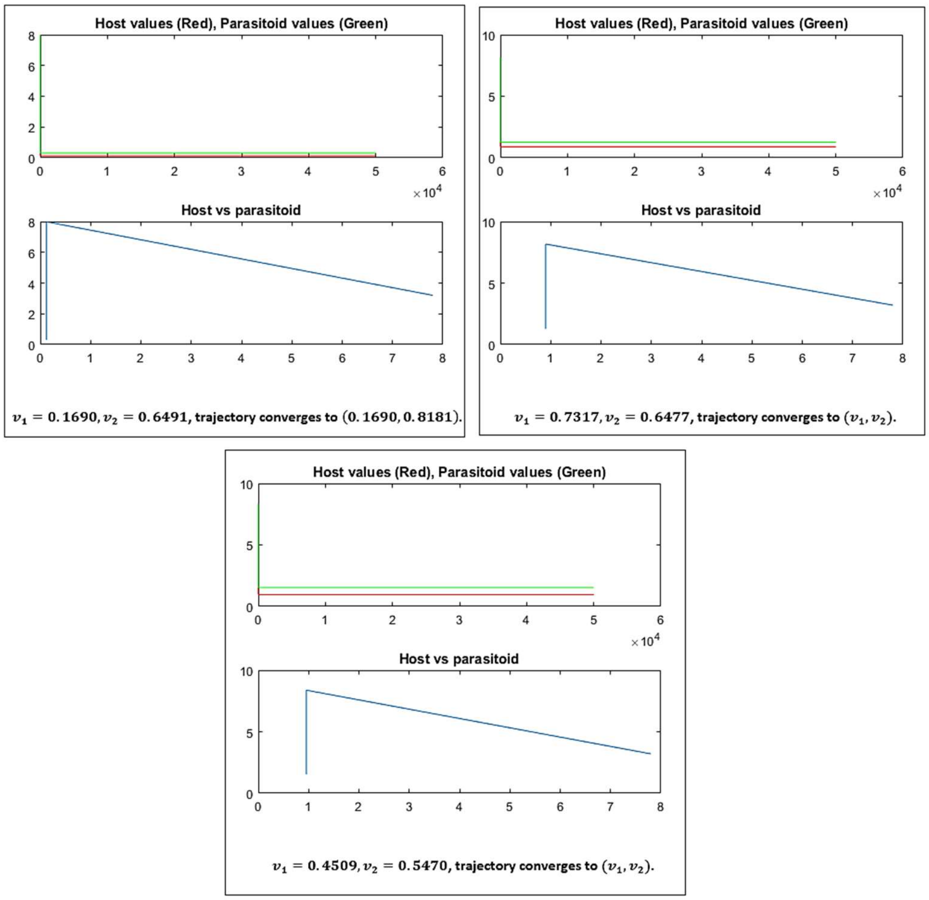


Fig. 4.19.2. Solution of Nicholson-Bailey model with uniformly distributed noise Eqn. (4.14) corresponding to the $(r, a, k, x, y) = (2.69, 32.72, 21.14, 7.8, 3.2)$ of the Nicholson-Bailey model Eqn. (4.1).

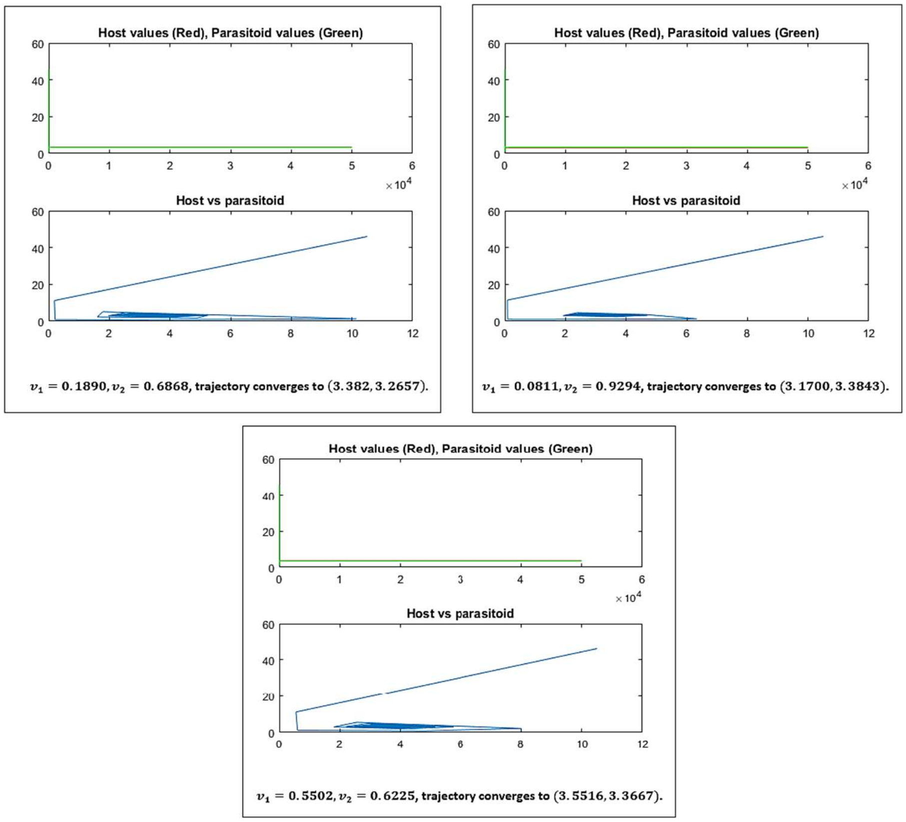


Fig. 4.19.3. Solution of Nicholson-Bailey model with uniformly distributed noise Eqn. (4.14) corresponding to the $(r, a, k, x, y) = (2.71, 0.44, 6.89, 10.5, 46)$ of the Nicholson-Bailey model Eqn. (4.1).

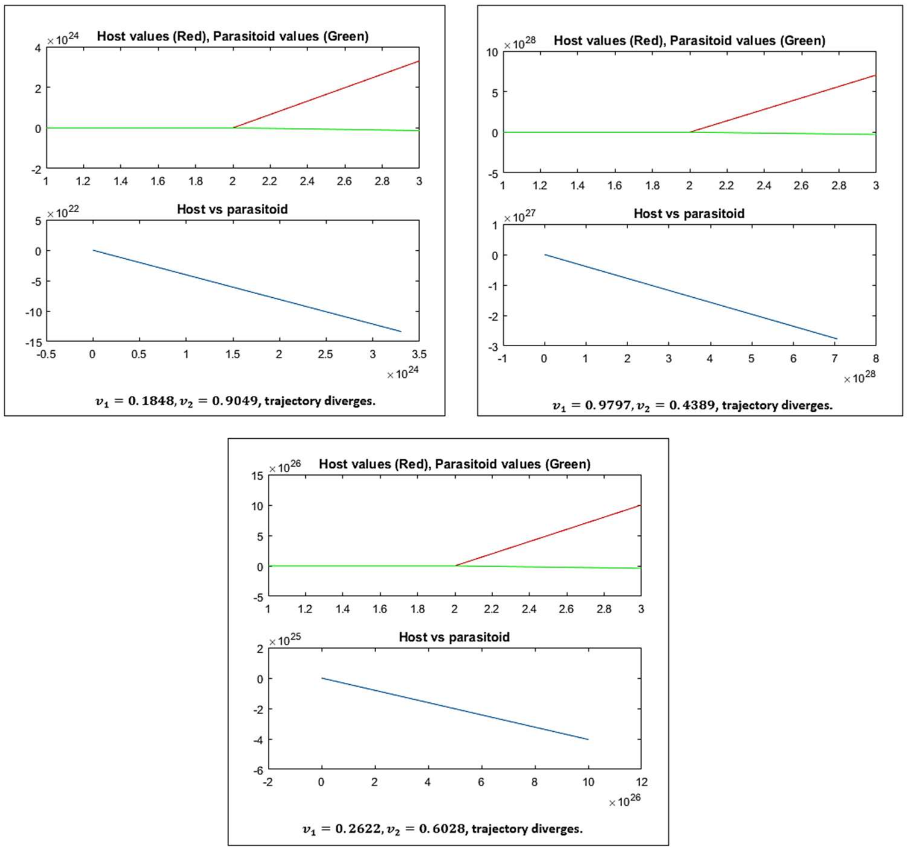


Fig. 4.19.4. Solution of Nicholson-Bailey model with uniformly distributed noise Eqn. (4.14) corresponding to the $(r, a, k, x, y) = (2.71, 0.44, 6.89, 10.5, 46)$ of the Nicholson-Bailey model Eqn. (4.1).

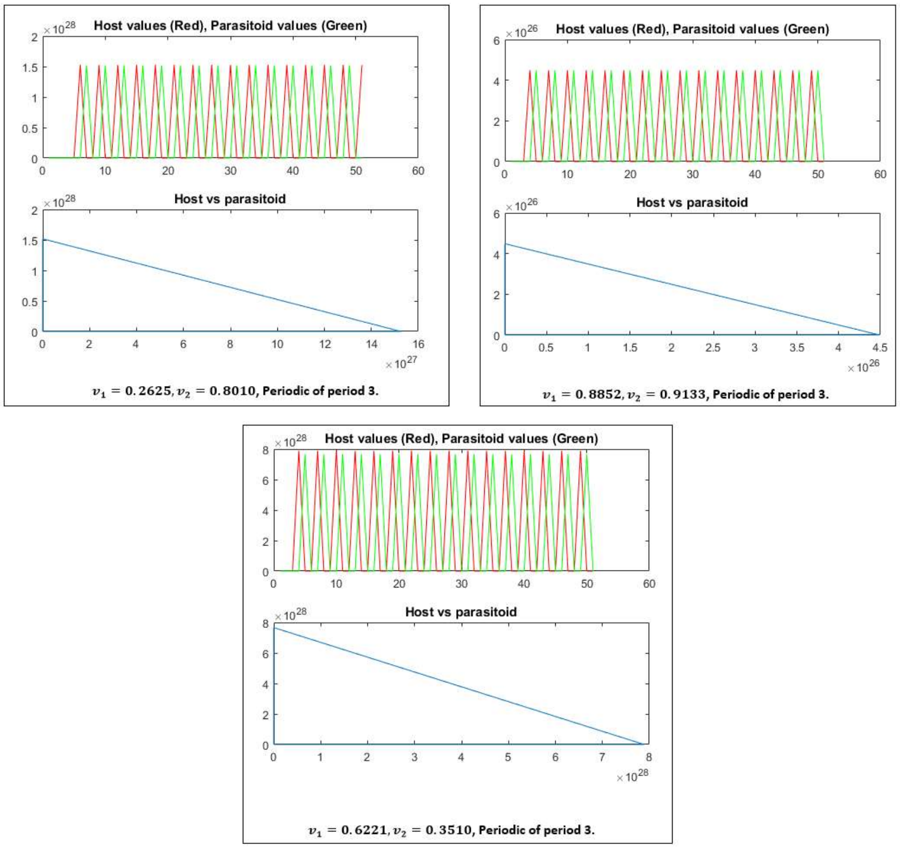


Fig. 4.20.1. Periodic solution of the Nicholson-Bailey model Eqn. (4.14) with uniformly distributed noise corresponding to the $(r, a, k) = (72, 5, 42)$.

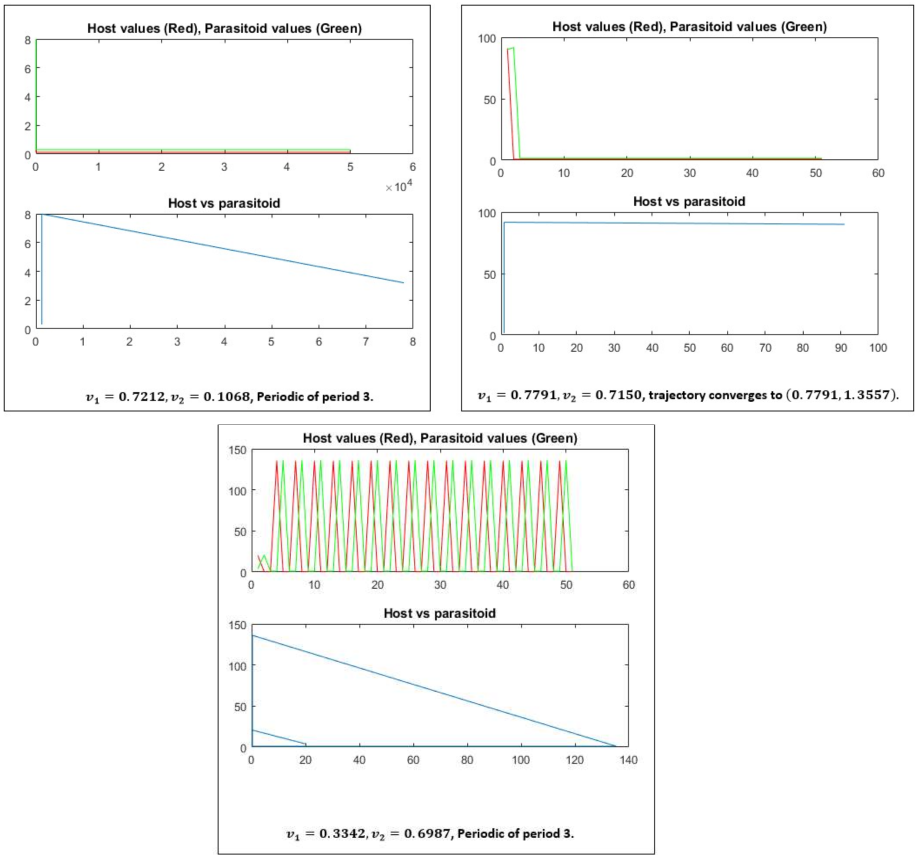


Fig. 4.20.2. Periodic solution of the Nicholson-Bailey model Eqn. (4.14) with uniformly distributed noise corresponding to the $(r, a, k) = (76, 67, 32)$.

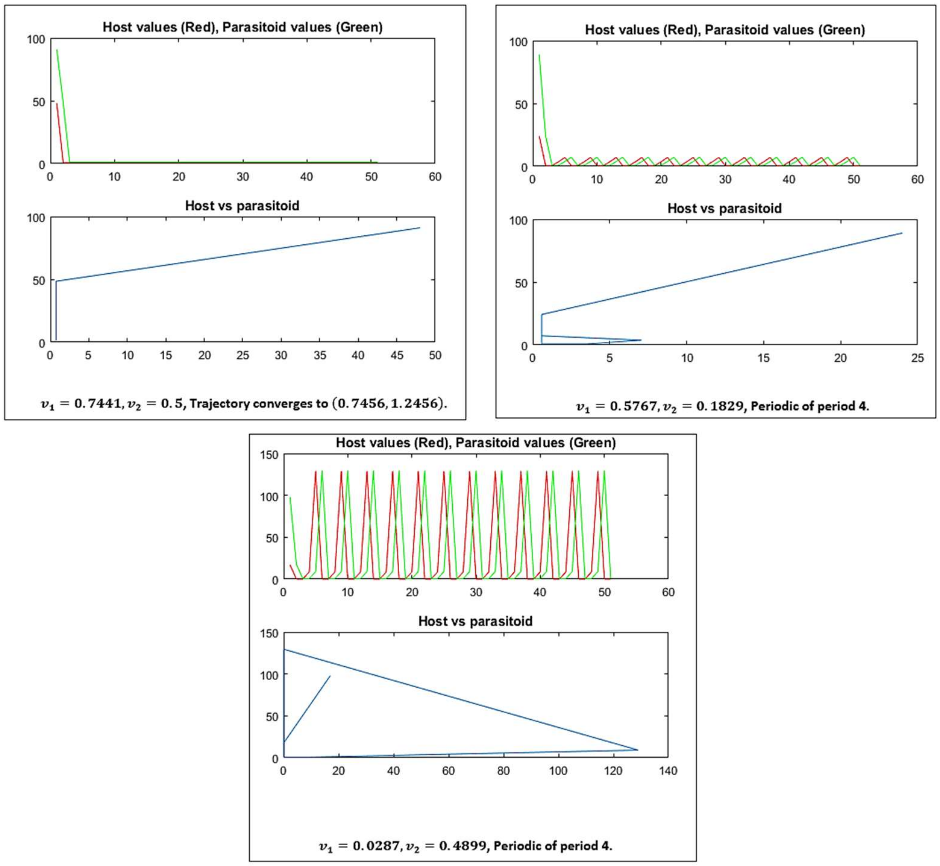


Fig. 4.20.3. Periodic solution of the Nicholson-Bailey model Eqn. (4.14) with uniformly distributed noise corresponding to the $(r, a, k) = (14, 16, 40)$.

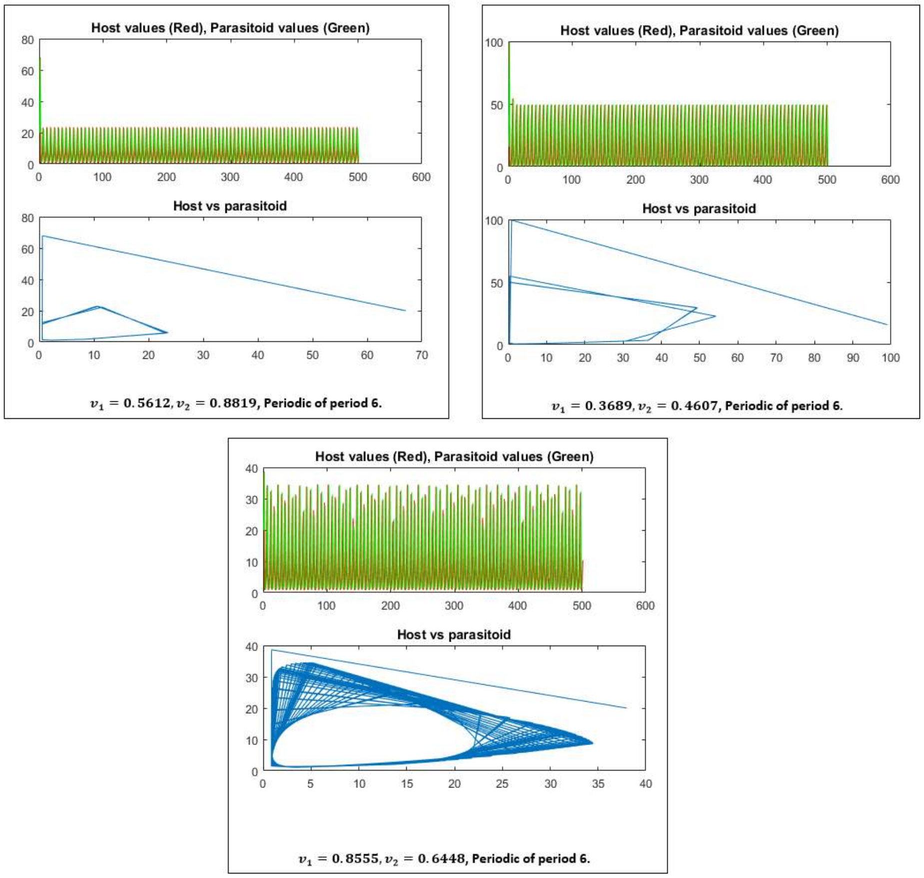


Fig. 4.20.4. Periodic solution of the Nicholson-Bailey model Eqn. (4.14) with uniformly distributed noise corresponding to the $(r, a, k) = (1.78328, 0.457574, -682)$.

4.8.4 Chaotic Solutions

Here, the existence of chaotic solutions of the noisy model Eqn. (4.14) corresponding to the parameters and initial values in Fig. 4.17.1 – Fig. 4.17.5 is discussed.

The study observes that the chaotic solutions of the noisy model Eqn. (4.14) with those sets of parameters and initial values for which the Nicholson-Bailey model Eqn. (4.14) possessed chaotic solutions, once the noise is very close to zero. It is found that with similar parameter and initial values, the trajectory behavior got changed due to its noise which is clearly depicted computationally in Fig. 4.21.1. – Fig. 4.21.4.

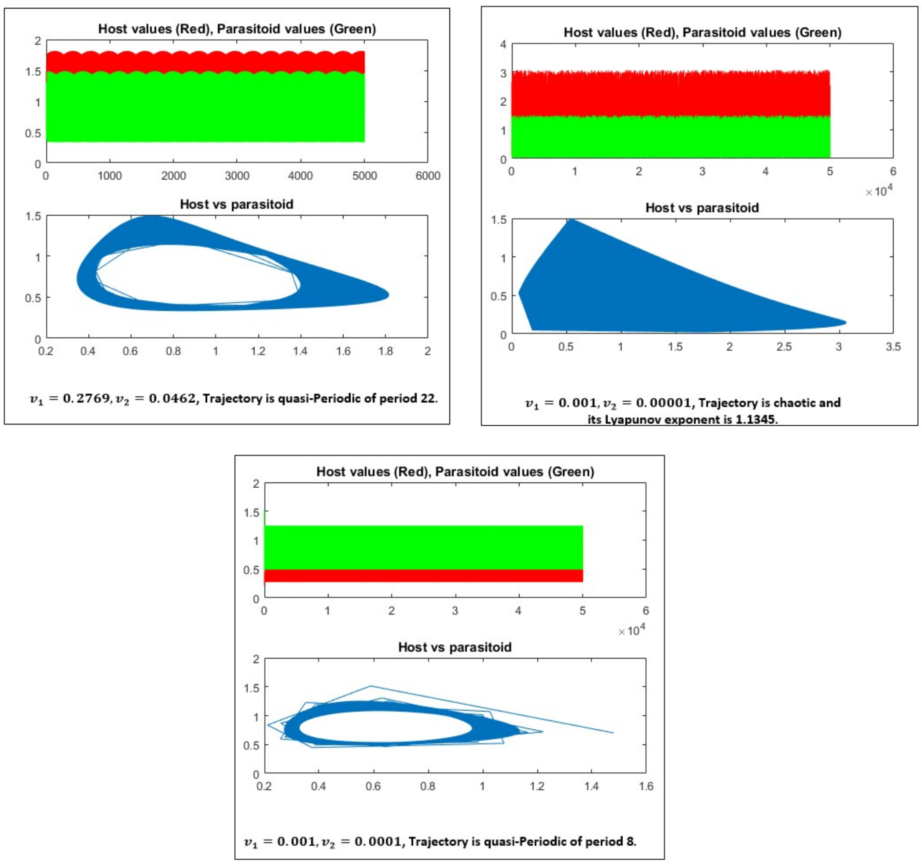


Fig. 4.21.1. Solution of Nicholson-Bailey model with Gaussian noise to Eqn. (4.14) corresponding to the parameters $(r, a, k, x, y) = (2.6, 2.7, 2, 1.48, 0.7)$ of the Nicholson-Bailey model Eqn. (4.1).

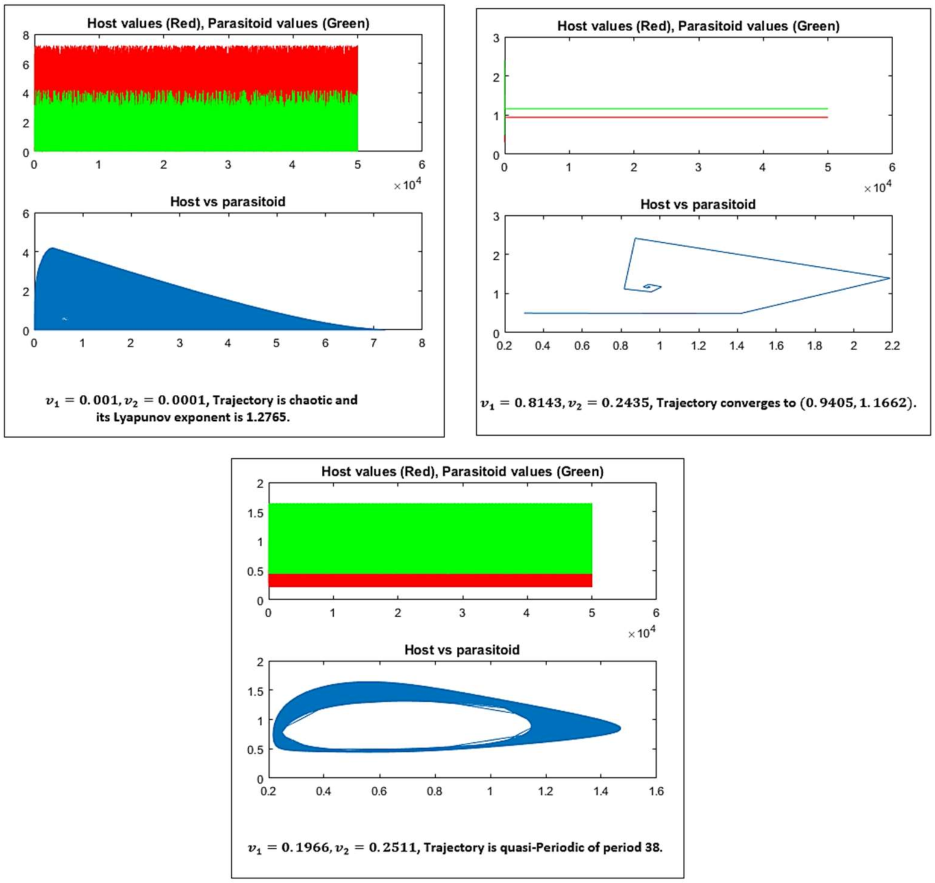


Fig. 4.21.2. Solution of Nicholson-Bailey model with Gaussian noise to Eqn. (4.14) corresponding to the parameters $(r, a, k, x, y) = (2.6, 2.7, 2, 1.48, 0.7)$ of the Nicholson-Bailey model Eqn. (4.1).

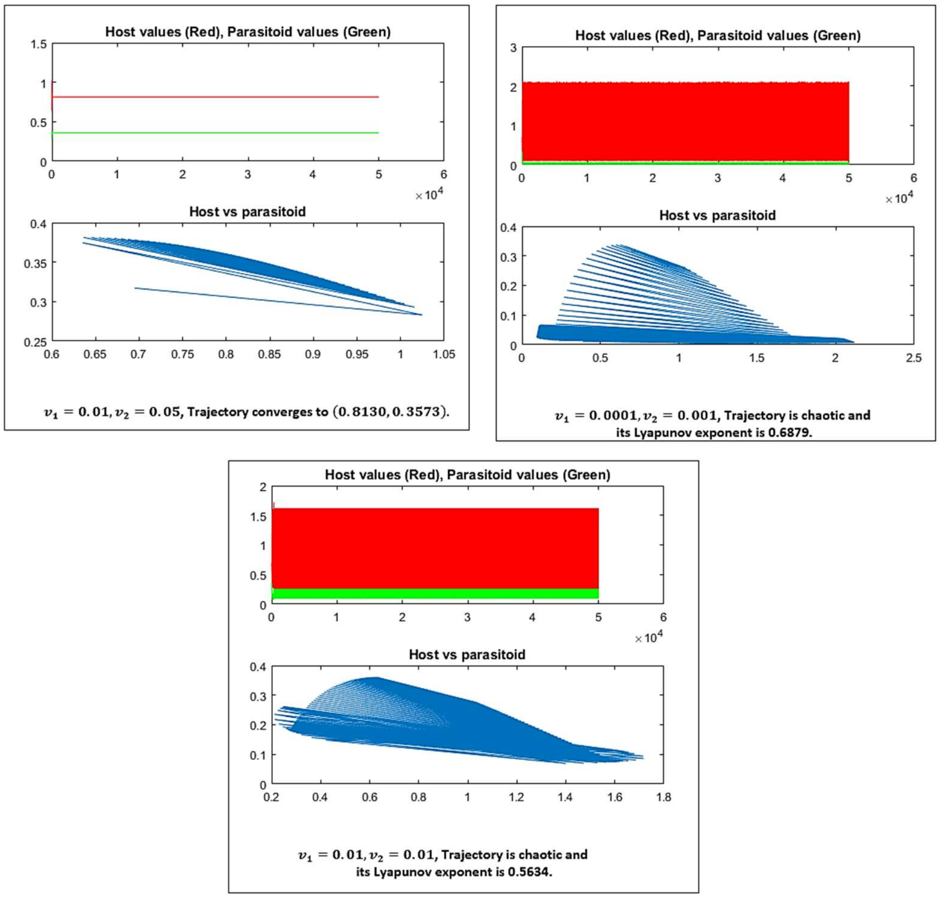


Fig. 4.21.3. Solution of Nicholson-Bailey model with Gaussian noise to Eqn. (4.14) corresponding to the parameters $(r, a, k, x, y) = (2.8, 1.5, 1, 0.6948, 0.3171)$ of the Nicholson-Bailey model Eqn. (4.1).

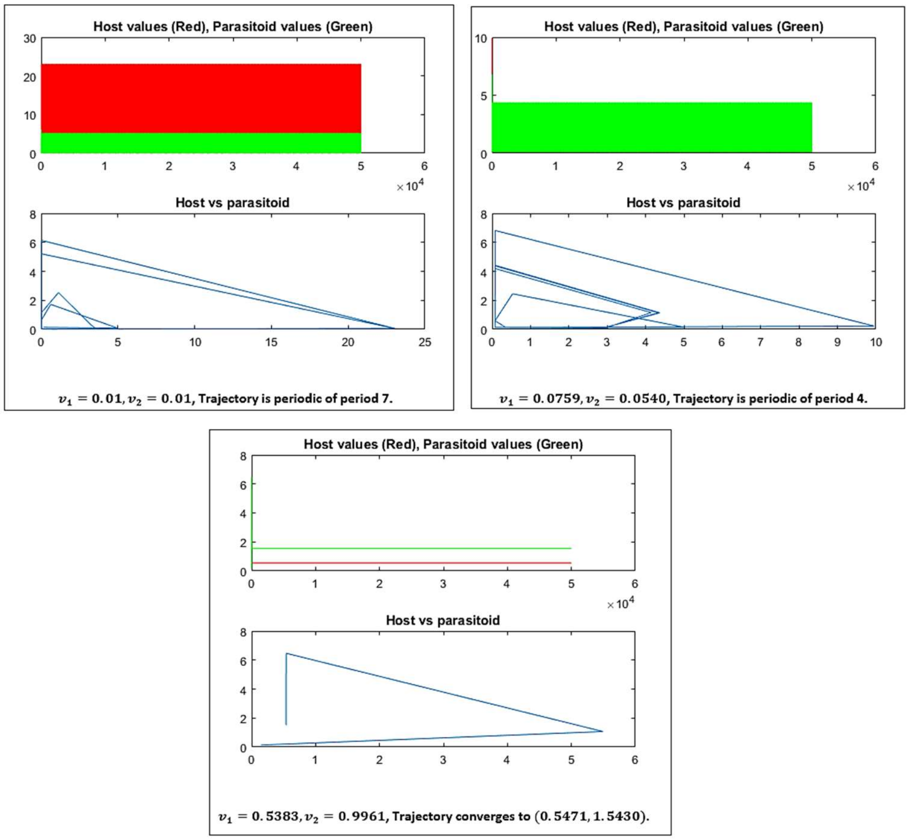


Fig. 4.21.4. Solution of Nicholson-Bailey model with Gaussian noise to Eqn. (4.14) corresponding to the parameters $(r, a, k, x, y) = (4.4, 5.1, 3.6, 0.1455, 0.1361)$ of the Nicholson-Bailey model Eqn. (4.1).

4.9 Conclusion

The dynamics of the classical Nicholson-Bailey system along with its scaled and noisy models were explored from the very mathematical abstraction, which might be applicable in understanding extended futuristic biological host parasitoid models. Over extending the domain of parameters to the entire real line from positive real numbers, the bilateral symmetry in dynamics (fixed points), parametric characteristics and scaling factors including several interesting results on periodicity and chaotic solutions of the models are studied.

CHAPTER 5. ROSENZWEIG-MACARTHUR PREDATOR-PREY MODELS

5.1 Introduction

In ecological dynamics, Rosenzweig-Macarthur predator – prey model is well studied since 1963 where all the parameters are considered as positive real numbers. In this study, by considering the parameters as complex numbers, the dynamics of the Rosenzweig-Macarthur model was investigated computationally. In addition, the model with different functional response from predator perspective is also studied and compared with the former one.

Inherent complexity of ecological systems makes them fascinating [103]. In these systems, the interactions among two individual species could be amazingly complex [104]. Apprehending dynamics of such interactions is considerably challenging because of complex behavior. In classical models, the dynamics of interacting species is explained by a system of ODEs [58], [105], [106]. Indeed, several important theoretical works in ecology presents derivations of functional forms that include individual level effects [92]. These systems are too complex for mathematical analysis even if the knowledge of interactions among species is available in detail and the corresponding mathematical expressions are obtained. [107], [108].

A well-known continuous – time predator-prey Rosenzweig-Macarthur model (a little variant) with functional response taken as Holling type II is described as following system of equations [27], [28], [109]–[112].

$$\frac{dx}{dt} = x\left(1 - \frac{x}{k}\right) - \frac{mxy}{1+x} \qquad \text{Eqn. (5.1)}$$

$$\frac{dy}{dt} = -cy + \frac{mxy}{1+x} \qquad \text{Eqn. (5.2)}$$

where x, k and m are positive real numbers. The variables x is prey population density and y is predator population density [106]. Mathematicians are always curious to explore what happens if we consider any real parameters instead of just positive real numbers. In appreciating such interest, the study considers the parameters c, k and m as arbitrary real numbers to understand if there is any hidden symmetry in discrete-time dynamical system. Consideration of negative parameters in the model would possibly loose biological direct implications but certainly this mathematical model would be mathematically complex.

The discrete version of the model is given by

$$x_{t+1} = x_t + \left(x_t\left(1 - \frac{x_t}{k}\right) - \frac{mx_ty_t}{1 + x_t}\right)dt \qquad \text{Eqn. (5.3)}$$

$$y_{t+1} = y_t + \left(-cy_t + \frac{mx_ty_t}{1 + x_t}\right)dt \qquad \text{Eqn. (5.4)}$$

where all the parameters c, k and m are real numbers and dt is the delay term. In the above equations Eqn. (5.3) and Eqn. (5.4), the functional response is considered as Holling type II from prey-perspective (variable x is associated to the functional response). The study hypothesizes another analogue model as defined under with functional response as Holling type II from predator perspective. The equations are

$$x_{t+1} = x_t + \left(x_t\left(1 - \frac{x_t}{k}\right) - \frac{mx_ty_t}{1 + y_t}\right)dt \qquad \text{Eqn. (5.5)}$$

$$y_{t+1} = y_t + \left(-cy_t + \frac{mx_ty_t}{1 + y_t}\right)dt \qquad \text{Eqn. (5.6)}$$

5.2 Computational dynamics of Equations (5.3-5.4)

5.2.1 Local Stability Analysis

This section presents one of the most well-known result in analyzing the local asymptotic stability of a fixed point before proceeding to do so for the fixed points of the discrete time Rosenzweig-Macarthur model equations Eqn. (5.3-5.4) [113]–[115].

Result 5.1: Let $(\bar{x}, \bar{y})$ be a fixed point of the system

$$x_{t+1} = f(x_t, y_t)$$
$$y_{t+1} = f(x_t, y_t)$$

Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ be the Jacobian at the point $(\bar{x}, \bar{y})$ with eigenvalues λ_1 and λ_2. Then,

- $|\lambda_{1,2}| < 1 \Longrightarrow (\bar{x}, \bar{y})$ is locally asymptotically stable or attracting.
- $|Tr(A)| < 1 + Det(A) < 2$, then $(\bar{x}, \bar{y})$ is locally asymptotically stable or attracting.
- $|\lambda_j| > 1$ for one $j = \{1,2\} \Longrightarrow (\bar{x}, \bar{y})$ is repelling.
- $|\lambda_j| = 1$ for one $j = \{1,2\} \Longrightarrow (\bar{x}, \bar{y})$ is saddle.
- The fixed points $(\bar{x}, \bar{y})$ of the Rosenzweig – Macarthur model Eqs (5.3-5.4) are solutions of the system of equations
- $\bar{x} = \bar{x} + \left(\bar{x}\left(1 - \frac{\bar{x}}{k}\right) - \frac{m\bar{x}\bar{y}}{1+\bar{x}}\right) dt$ Eqn. (5.7)
- $\bar{y} = \bar{y} + \left(-c\bar{y} + \frac{m\bar{x}\bar{y}}{1+\bar{x}}\right) dt$ Eqn. (5.8)

The system of equations (5.7-5.8) give three unique fixed points $(k, 0), (0,0)$, and $\left(-\frac{c}{c-m}, \frac{c(-k)-c+km}{k(c-m)^2}\right)$. The locally asymptotic stability analysis of the fixed points by making the system of equations Eqn. (5.3-5.4) linearized about the fixed points is discussed in the following subsections.

Local stability analysis of $(\boldsymbol{k}, \boldsymbol{0})$. The linearized system $X_{t+1} = JX_t$ (where $X_t = [x_t, y_t]^T$ and J is the Jacobian) is obtained by linearizing the model equations Eqn. (5.3-5.4) about the fixed point $(k, 0)$. The Jacobian about the $(k, 0)$ is

$$J_{(k,0)} = \begin{pmatrix} 1 - dt & -\dfrac{dtkm}{k+1} \\ 0 & dt\left(\dfrac{km}{k+1} - c\right) + 1 \end{pmatrix}$$

For $J_{(k,0)}$, the eigenvalues are $1 - dt$ and $\frac{-cdtk-cdt+dtkm+k}{k+1}$.

The attracting, repelling and saddle conditions for the fixed point $(k, 0)$ are given in the following tables Table 5.1-Table 5.3 respectively.

Table 5.1. Eigenvalues and corresponding attracting conditions of $(k, 0)$ for different delays.

Delay Term (dt)	Eigenvalues	Conditions
1	0 & $\frac{-ck-c+km+k+1}{k+1}$	$0 < c - \frac{km}{k+1} < 2$
0.5	0.5 & $\frac{-0.5ck-0.5c+0.5km+k+1}{k+1}$	$\frac{km}{k+1} < c < \frac{k(m+4)+4}{k+1}$
0.005	0.995 & $\frac{-0.005ck-0.005c+0.005km+k+1}{k+1}$	$\frac{km}{k+1} < c < \frac{k(m+400)+400}{k+1}$

Table 5.2. Eigenvalues and corresponding repelling conditions of $(k, 0)$ for different delays.

Delay Term (dt)	Eigenvalues	Conditions
1	0 & $\frac{-ck-c+km+k+1}{k+1}$	$c < \frac{km}{k+1}, k+1 \neq 0$
0.5	0.5 & $\frac{-0.5ck-0.5c+0.5km+k+1}{k+1}$	$c < \frac{km}{k+1}, k+1 \neq 0$
0.005	0.995 & $\frac{-0.005ck-0.005c+0.005km+k+1}{k+1}$	$c < \frac{km}{k+1}, k+1 \neq 0$

Table 5.3. Eigenvalues and corresponding saddle conditions of $(k, 0)$ for different delays.

Delay Term (dt)	Eigenvalues	Conditions
1	0 & $\frac{-ck - c + km + k + 1}{k + 1}$	$c = \frac{2 + km}{k + 1}, k + 1 \neq 1$
0.5	0.5 & $\frac{-0.5ck - 0.5c + 0.5km + k + 1}{k + 1}$	$c = \frac{4 + k(4 + m)}{k + 1}, k + 1 \neq 1$
0.005	0.995 & $\frac{-0.005ck - 0.005c + 0.005km + k + 1}{k + 1}$	$c = \frac{k(m + 400) + 400}{k + 1}, k + 1 \neq 1$

Here the visualizations three-dimensional subspaces $S_{(attracting,dt)}, S_{(repelling,dt)}$, and $S_{(saddle,dt)}$ of $\mathbb{R}^3$ for different delays dt, which are shown in Fig 5.1.1-Fig.5.1.3 In precise, $S_{(attracting,dt)}$ denotes the space of parameters (c, k, m) in $\mathbb{R}^3$ for which the fixed point $(k, 0)$ is attracting and similarly others follow.

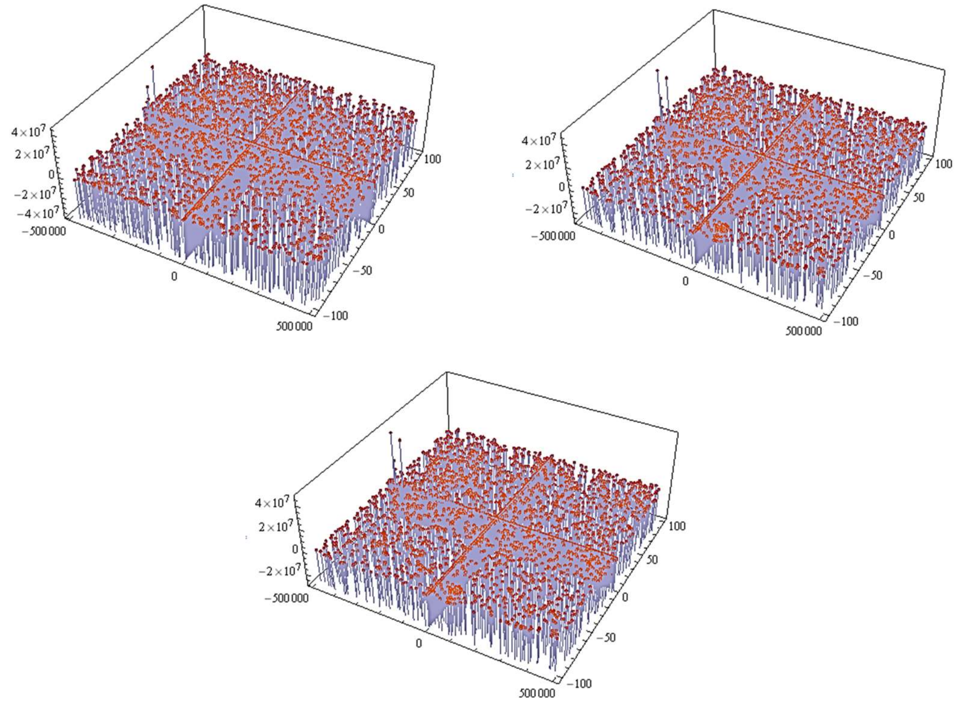


Fig 5.1.1 Top Left: $S_{(attracting,1)}$, **Top Right:** $S_{(attracting,0.5)}$, **Bottom:** $S_{(attracting,0.005)}$.

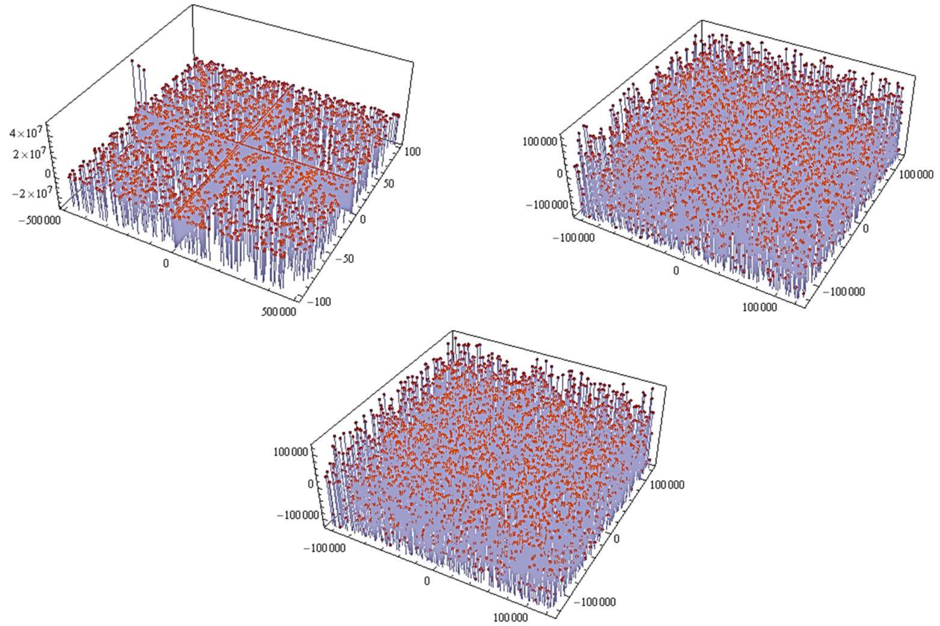


Fig 5.1.2 Top Left: $S_{(repelling,1)}$, **Top Right:** $S_{(repelling,0.5)}$, **Bottom:** $S_{(repelling,0.005)}$.

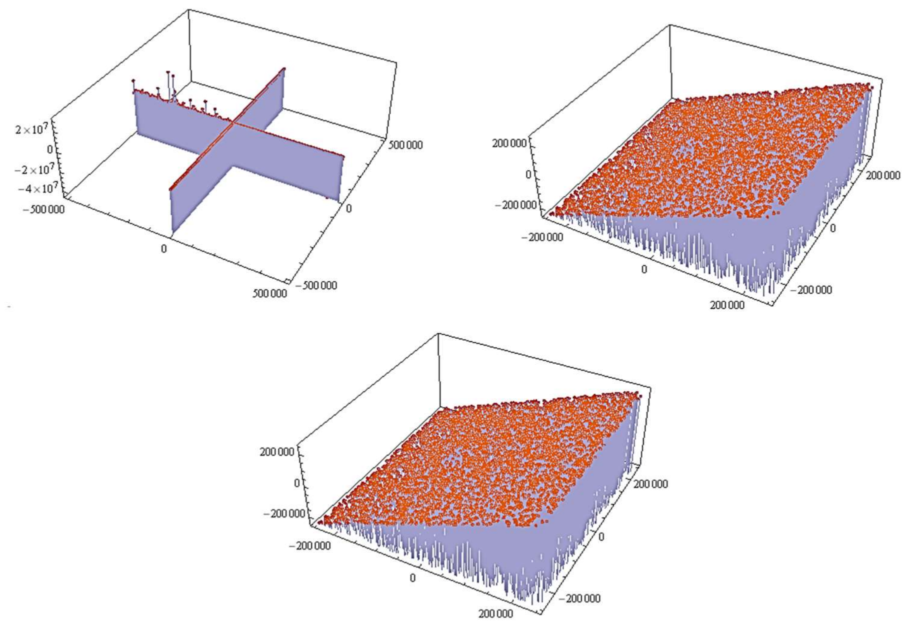


Fig 5.1.3 Top Left: $S_{(saddle,1)}$, Top Right: $S_{(saddle,0.5)}$, Bottom: $S_{(saddle,0.005)}$.

The following Table 5.4 demonstrates a couple of examples of parameters c, k, and m for which the fixed point $(k, 0)$ is attracting.

Table 5.4. Examples of parameters where $(k, 0)$ is attracting for different delays.

Delay Term (dt)	Parameters (c, k, m)	Nature
1	$(0.5219, -0.2843, 2.2381)$	Attracting to $(-0.2843, 0)$
1	$(-37, 56, -38.48)$	Attracting to $(56, 0)$
0.5	$(3.84127, -10, -0.137931)$	Attracting to $(-10, 0)$
0.5	$(321, -82, 315.846)$	Attracting to $(-82, 0)$
0.005	$(-106.9, -115.9, 57)$	Attracting to $(-115.9, 0)$
0.005	$(31.4, 39, 21)$	Attracting to $(39, 0)$

The trajectory plot is depicted in the following Fig 5.2.1-Fig 5.2.6 for each of the examples given in Table 5.4.

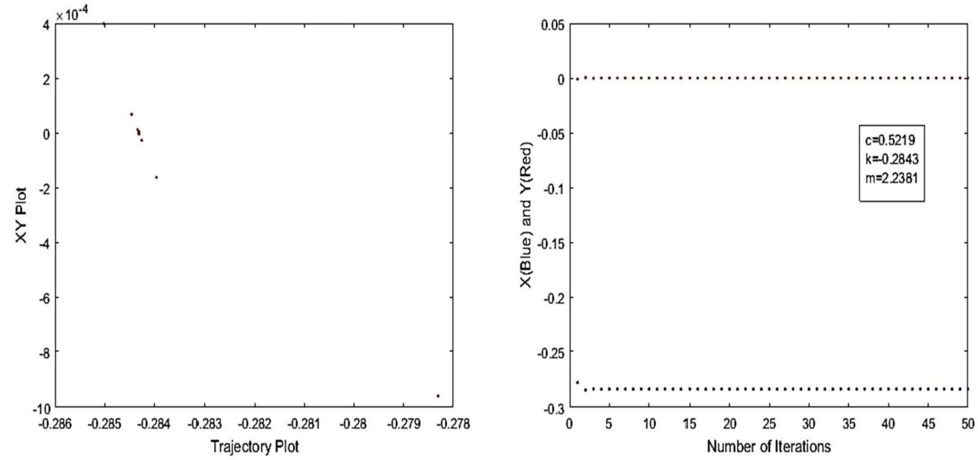


Figure 5.2.1. Examples of trajectories where the fixed point $(k, 0)$ is attracting for $(c, k, m) = (0.5219, -0.2843, 2.2381)$.

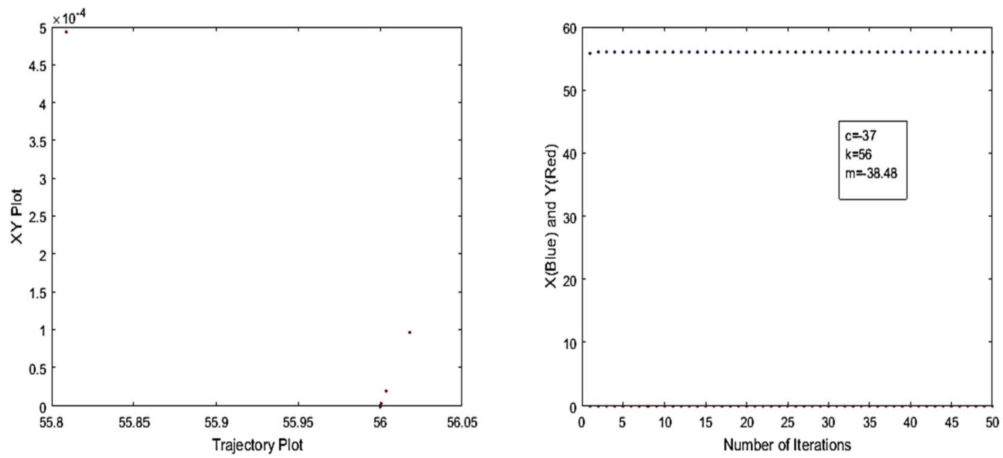


Figure 5.2.2. Examples of trajectories where the fixed point $(k, 0)$ is attracting for $(c, k, m) = (-37, 56, -38.48)$.

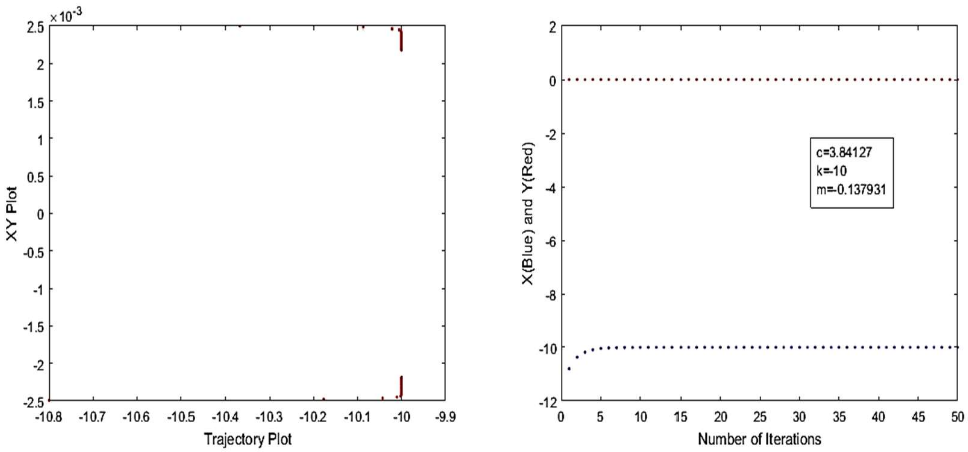


Figure 5.2.3. Examples of trajectories where the fixed point $(k, 0)$ is attracting for $(c, k, m) = (3.84127, -10, -0.137931)$.

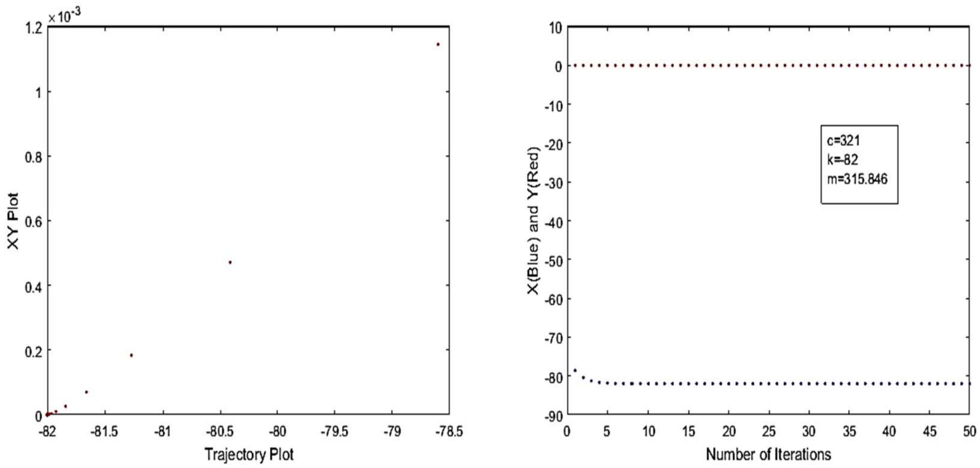


Figure 5.2.4. Examples of trajectories where the fixed point $(k, 0)$ is attracting for $(c, k, m) = (321, -82, 315.846)$.

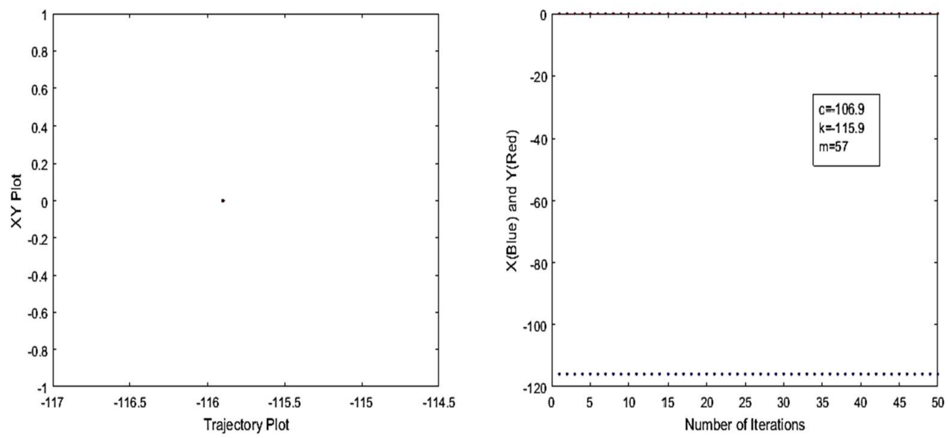


Figure 5.2.5. Examples of trajectories where the fixed point $(\boldsymbol{k}, \mathbf{0})$ is attracting for $(\boldsymbol{c}, \boldsymbol{k}, \boldsymbol{m}) = (-\mathbf{106.9}, -\mathbf{115.9}, \mathbf{57})$.

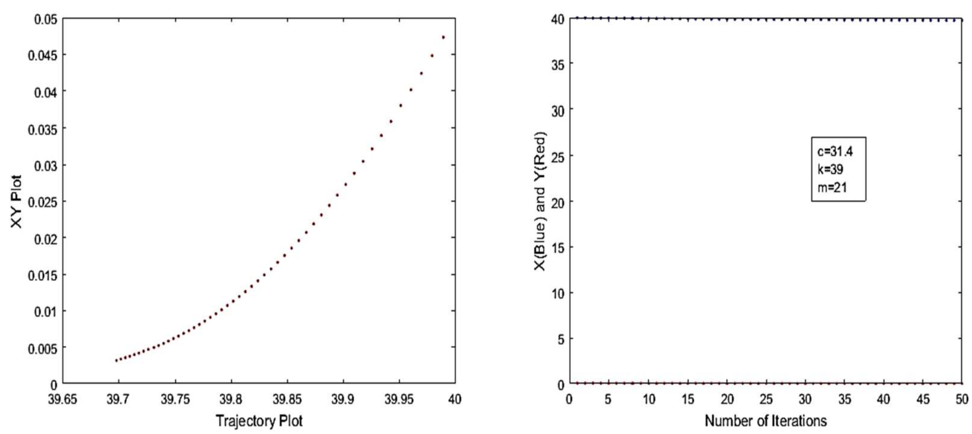


Figure 5.2.6. Examples of trajectories where the fixed point $(\boldsymbol{k}, \mathbf{0})$ is attracting for $(\boldsymbol{c}, \boldsymbol{k}, \boldsymbol{m}) = (\mathbf{31.4}, \mathbf{39}, \mathbf{21})$.

Now an example was given for parameters $(c = 16, k = 139, m = 182)$ and delay $dt = 0.005$ where the fixed point $(k, 0)$ is repelling as shown in the Fig 5.3.

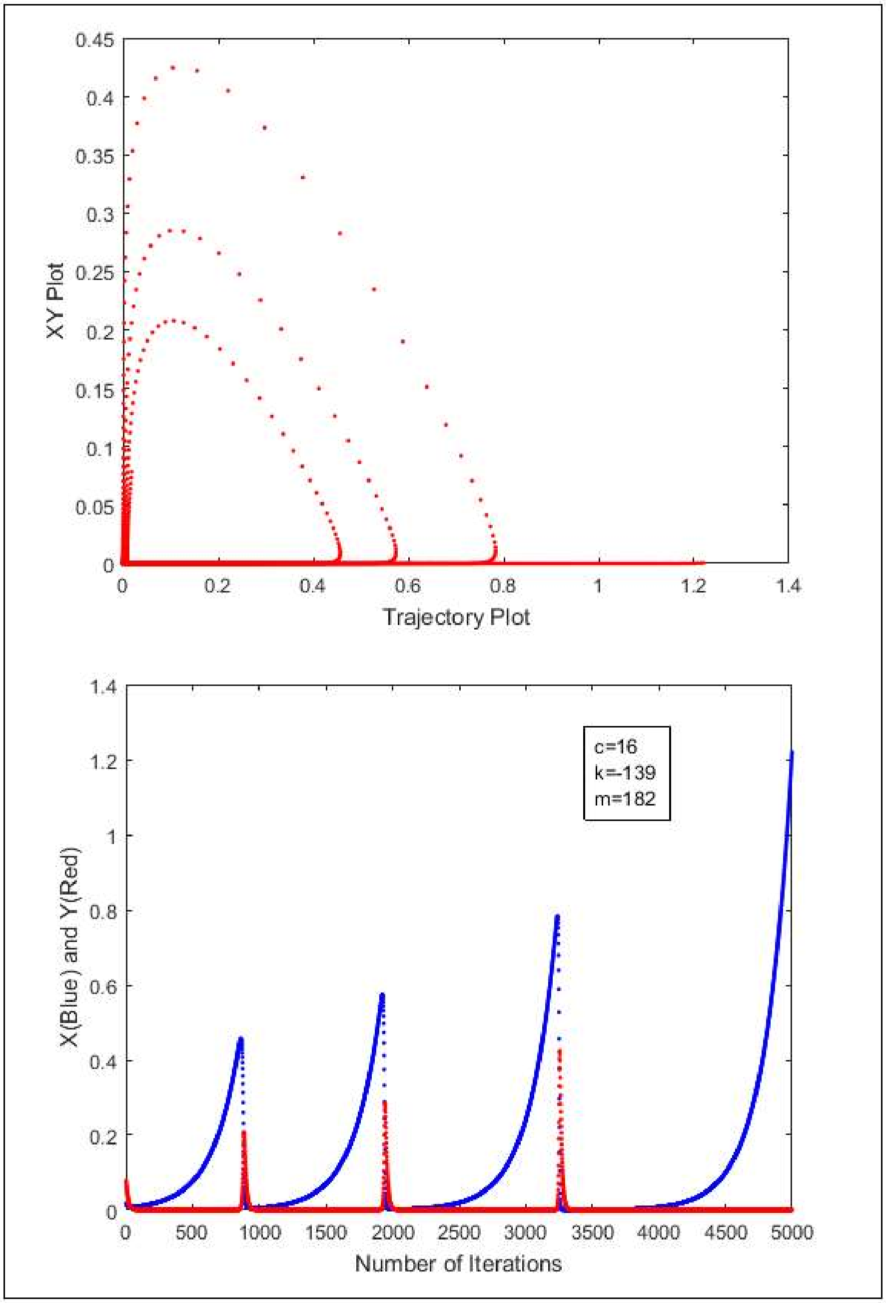


Figure 5.3. Example of trajectory where the fixed point $(k, 0)$ is repelling.

The fixed point $(k, 0)$ exhibits saddle point behavior for delay $dt = 0.005$ and $c = -44.39, k = 203.2,$ and $m = -44.6$ for 10 distinct sets of initial values as depicted in Fig 5.4.

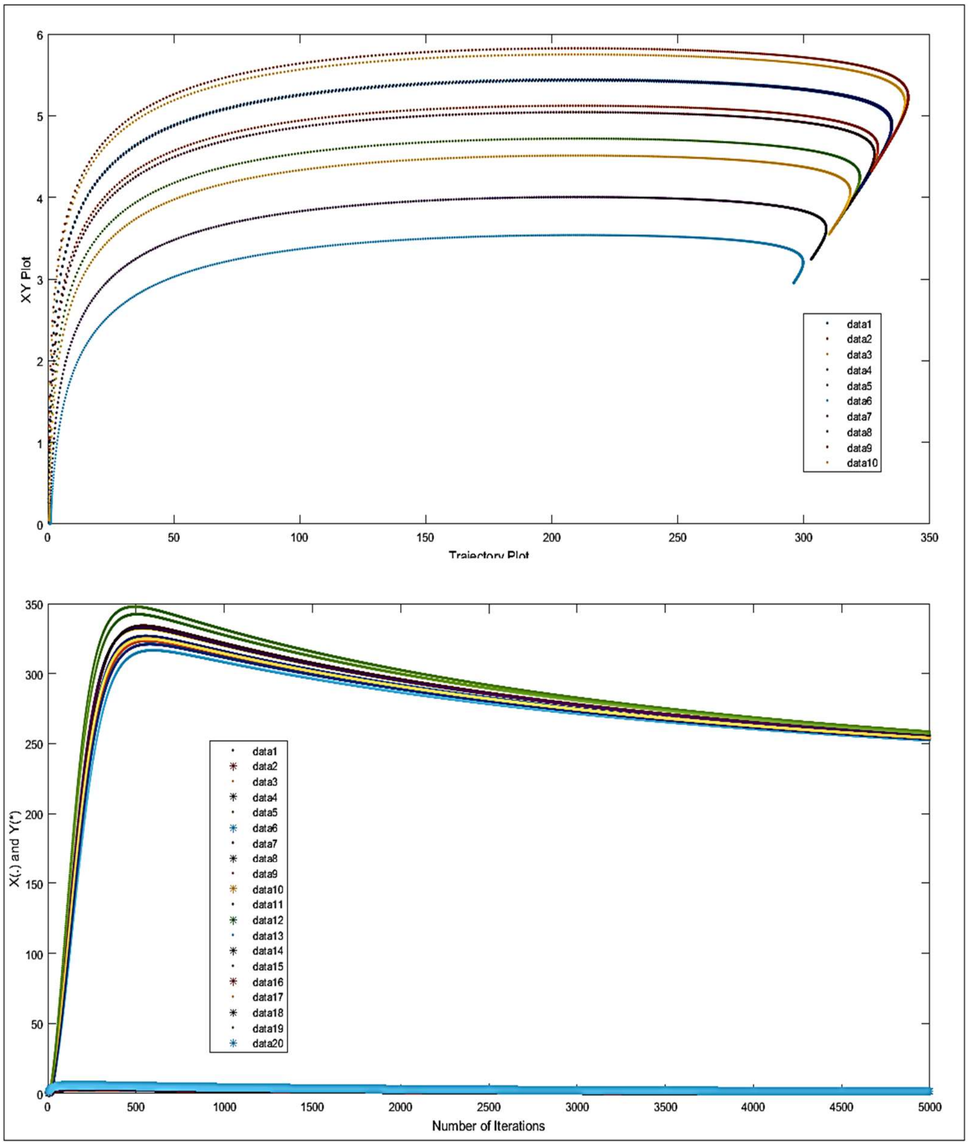


Figure 5.4. Example of trajectories where the fixed point $(k, 0)$ is saddle.

Local stability analysis of $(\mathbf{0},\mathbf{0})$: The Jacobian around (0,0) is given by

$$J_{(0,0)} = \begin{pmatrix} 1+dt & 0 \\ 0 & 1-cdt \end{pmatrix}$$

The eigenvalues of the Jacobian matrix $J_{(0,0)}$ are $1+dt$ and$1-cdt$. Therefore for any delay term $dt > 0$, there does not exist any parameter such that the absolute value of both the eigenvalues are less than one. Hence the fixed point $(0,0)$ is never be attracting.

Theorem 5.1: The fixed point (0,0) of the model Eqn. (5.3-5.4) is repelling for $dt > 0$ and $c > 0$, whenever $cdt < 2$.
Proof. The proof is straightforward from the result.

Here presented three examples with different parameters (Table 5.5) with varied delay term dt and corresponding trajectories as shown in Figures Fig. 5.5.1-Fig. 5.5.3 where the fixed point $(0,0)$ is repelling.

Table 5.5. Examples of parameters where $(\mathbf{0},\mathbf{0})$ is repelling for different delays $\mathbf{0.5}, \mathbf{0.005}$ and $\mathbf{0.0005}$ respectively.

Delay Term (dt)	**Parameters (c,k,m)**	**Nature**
0.5	$(204, 65,4)$	Attracting to $(65,0)$ and it is repelling from (0,0) for 10 different initial values.
0.005	$(304,50,4)$	Attracting to $(50,0)$ and it is repelling from (0,0) for 10 different initial values.
0.0005	$(304,5,5)$	Attracting to $(5,0)$ and it is repelling from (0,0) for 10 different initial values.

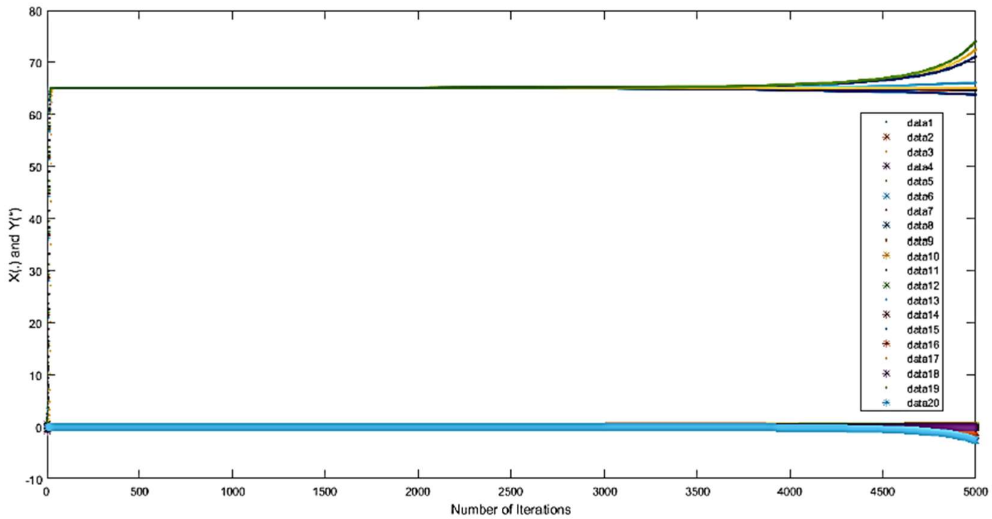


Fig 5.5.1 Examples of trajectories where fixed point $(0, 0)$ is repelling with different delay $dt = 0.5$.

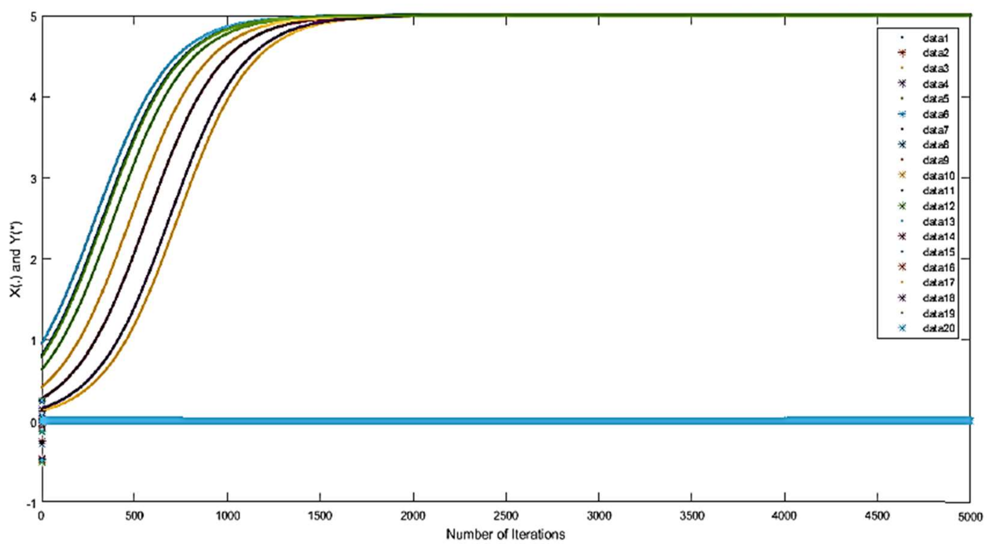


Fig 5.5.2 Examples of trajectories where fixed point $(0, 0)$ is repelling with different delay $dt = 0.05$.

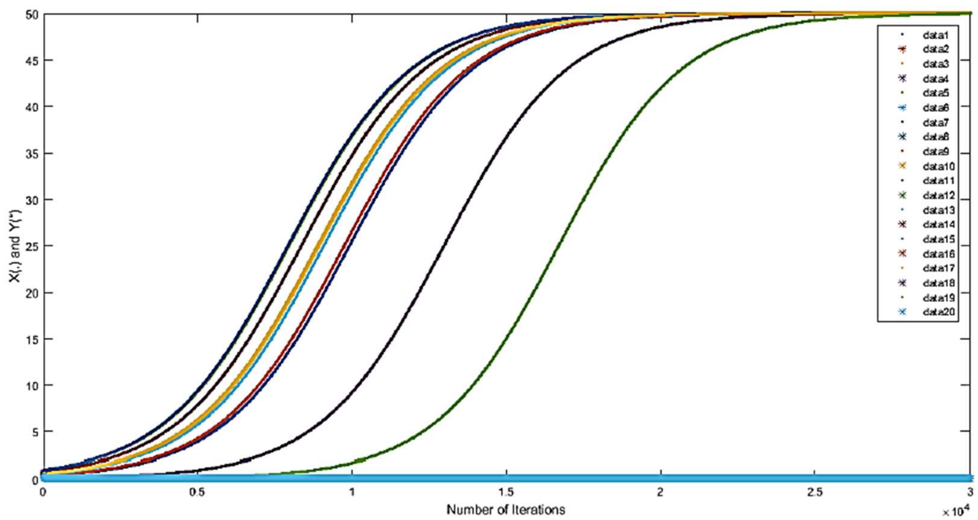


Fig 5.5.3 Examples of trajectories where fixed point $(\mathbf{0},\mathbf{0})$ is repelling with different delay $dt = 0.0005$.

The point (0,0) exhibits saddle point behavior for delay $dt = 0.005$ and $c = 0, k = 50$ and $m = 4$ for 10 different set of initial values as shown in the Fig. 5.6.

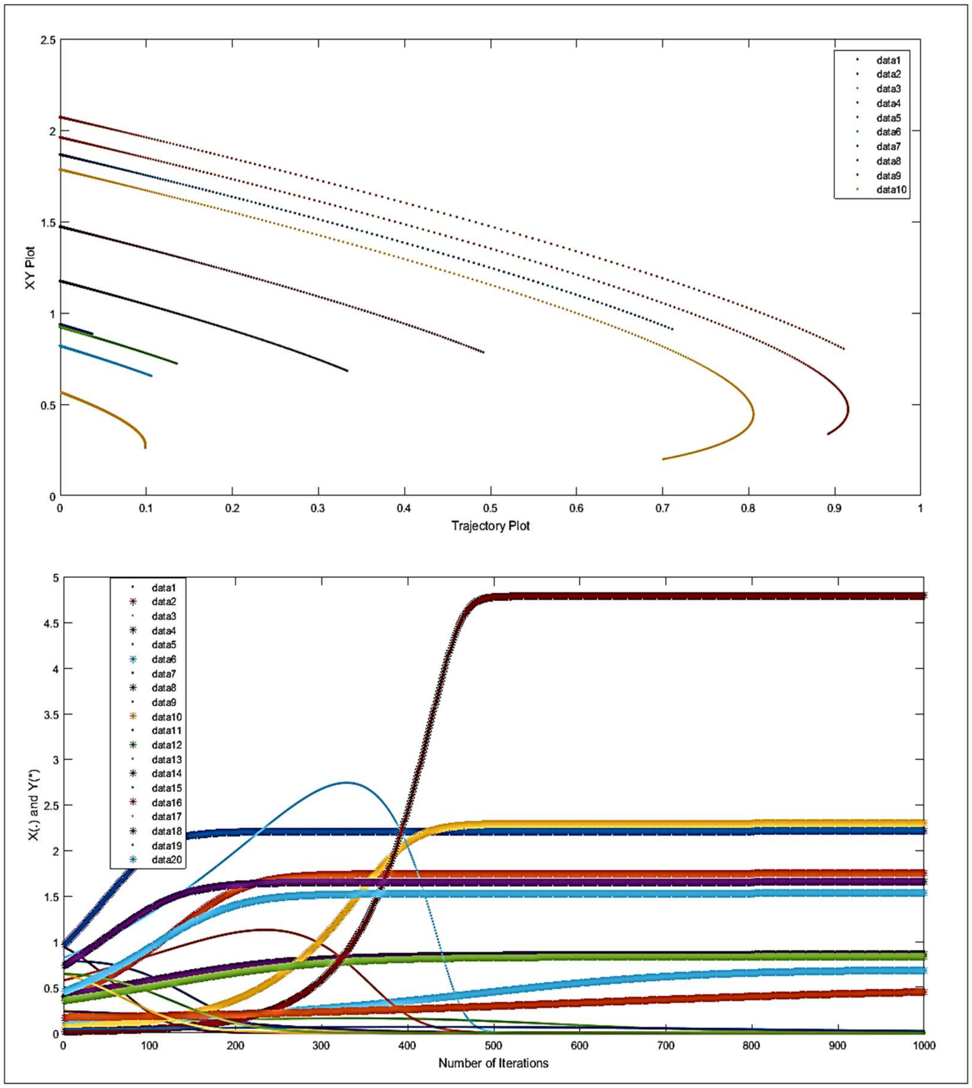


Fig 5.6. Examples of trajectories where the fixed point $(0, 0)$ is saddle.

Local stability analysis of $\left(-\frac{c}{c-m}, \frac{c(-k)-c+km}{k(c-m)^2}\right)$

The Jacobian is given by

$$\begin{pmatrix} \frac{c(k+1)dt}{km} + \frac{2cdt}{ck-km} + 1 & -cdt \\ -\frac{dt(kc+c-km)}{km} & 1 \end{pmatrix}$$

The determinant (D) and Trace (T) are given by

$cdt^2 - \frac{cdt(k+1)(cdt\quad)}{km} + \frac{2cdt}{ck} + 1$ and $\frac{cdt(k+1)}{km} + \frac{2cdt}{ck-km} + 2$ resepctively.

Then from the Result 5.1, the following theorem is proposed.

Theorem 5.2. The fixed point $\left(-\frac{c}{c-m}, \frac{c(-k)-c+km}{k(c-m)^2}\right)$ is attracting if $|\boldsymbol{T}| < \mathbf{1} + \boldsymbol{D} < \mathbf{2}$.

The three-dimensional subspaces $S_{(saddle,dt)}$ of $\mathbb{R}^3$ for different delays dt are shown in the following Fig 5.7.

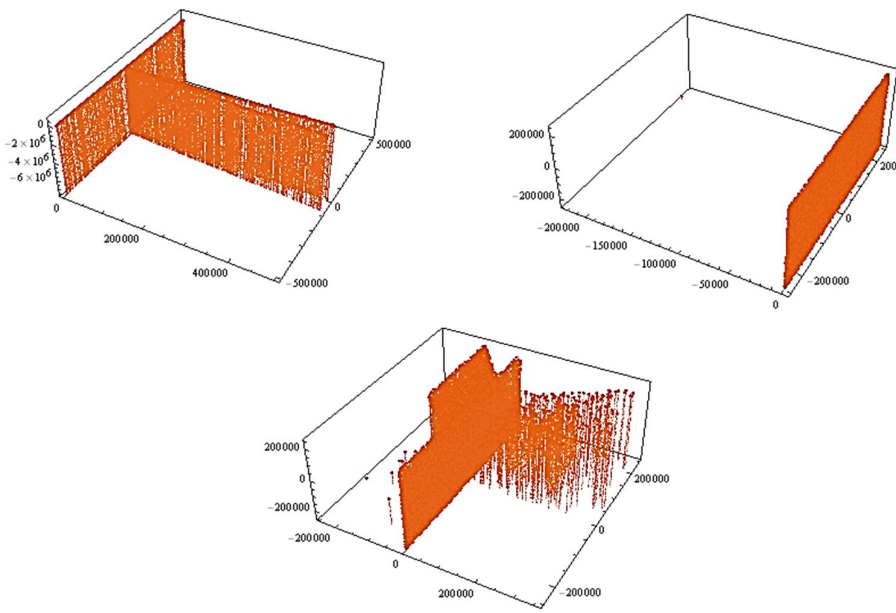


Fig 5.7. Top Left. $S_{(saddle,1)}$, Top Right: $S_{(saddle,0.5)}$, Bottom: $S_{(saddle,0.005)}$

For describing the conditions of fixed point's local stability, illustrations on the above theorem with different choices of sign of the parameters $c, k,$ and m are helpful.

Table 5.6. Local Asymptotical Stability (attracting) conditions for delay $dt = 1$

Parameters with Sign	Conditions for attracting
$c > 0, k > 0$ & $m > 0$	$c > 0, c < m, ck + c < km$ and $(c - m - 1)(ck + c - km) < m$
$c > 0, k > 0$ & $m < 0$	$0 < c < \frac{2}{3}\left(-5k + \sqrt{k(25k + 22 + 1} - 1\right),$ $\frac{-\sqrt{c^2 + c(6k+2) + (k-1)^2} + 2ck + c - k + 1}{2k} < m < \frac{c(-\sqrt{4(c+10)k + (c+2)^2 + 36k^2} + 2ck + c + 2k + 2)}{2(c+4)k}$
$c > 0, k < 0$ & $m < 0$	$m \leq -1, \frac{c-1}{c-m-1} + \frac{2c}{m-c} < k < \frac{c}{m-c}$
$c < 0, k < 0$ & $m < 0$	$c < m, m < (c - m - 1)(ck + c - km), km < ck + c$
$c < 0, k < 0$ & $m > 0$	$0 < m \leq 1, \frac{c}{c-m} + k > 0, 6c + \sqrt{9m(m+8) + 16} > 3m + 4,$ $k < \frac{c(c(-c + m + 2) + 2m)}{(c-m)((c-2)c - (c+4)m)}$
$c > 0, k < 0$ & $m > 0$	$m = 1, 1 < c < \frac{7 + \sqrt{97}}{6}, k < ck + c,$ $c\big(-c\big((c-4)k + c - 3\big) + k + 2\big) < 4k$
$c < 0, k > 0$ & $m < 0$	$3m + \sqrt{9m(m+8) + 16} + 4 \geq 6c, \frac{c}{c-m} + k < 0, \frac{c-1}{c-m-1} + \frac{2c}{m-c} < k$ $6c + \sqrt{9m(m+8) + 16} > 3m + 4, m + 4\sqrt{6} + 10 < 0$

Table 5.7. Local Asymptotic Stability (attracting) conditions for delay $dt = 0.5$

Parameters with Sign	Conditions for attracting
$c > 0, k > 0$ & $m > 0$	$0 < c < m, \frac{-c}{c-m} < k < \frac{c(-c+m+2)+2m}{c^2+c(-2m-2)+m(m+2)}$
$c > 0, k > 0$ & $m < 0$	$0 < c < -10.2857k\sqrt{k(81k+46)+1} - 1.14286,$ $\frac{-0.5\sqrt{c^2+12ck+4c+4k^2-8k+4}+ck+0.5c-k+1}{k} < m <$ $\frac{c(-0.5\sqrt{c(8-8k)+c^2+k(400k+352)+16}+c(k+0.5)+6k+2}{(c+16)k}$
$c > 0, k < 0$ & $m < 0$	$m \leq -2, \frac{c(-c+m+2)+2m}{c^2+c(-2m-2)+m(m+2)} < k < \frac{c}{m-c}$
$c < 0, k < 0$ & $m < 0$	$c < m, \frac{c(-c+m+2)+2m}{c^2+c(-2m-2)+m(m+2)} < k < \frac{c}{m-c}$
$c < 0, k < 0$ & $m > 0$	$\frac{c}{c-m} + k > 0, k < \frac{c(-c+m+2)+2m}{c^2+c(-2m-2)+m(m+2)},$ $c + 0.0714286\sqrt{m(49m+672)+256} \leq 0.5m + 1.14286$
$c > 0, k < 0$ & $m > 0$	$0 < m < 6, \frac{c}{c-m} + k > 0,$ $c > 0.5m + 0.0714286\sqrt{m(49m+672)+256} + 1.14286,$ $k < \frac{c(-c+m+2)+2m}{c^2+c(-2m-2)+m(m+2)}$
$c < 0, k > 0$ & $m < 0$	$m = -13.3221, c = -5.5182, 0.118847 < k < 0.707107$

Table 5.8. Local Asymptotic Stability (attracting) conditions for delay $dt = 0.005$.

Parameters with Sign	Conditions for attracting
$c > 0, k > 0$ & $m > 0$	$0 < c < m, \frac{-c}{c-m} < k < \frac{c(-c+m+200)+200m}{c^2+c(-2m-200)+m(m+200)}$
$c > 0, k > 0$ & $m < 0$	$0 < c < -80200.3k + 100.125\sqrt{k(641601k+4798)+1} - 100.125$, $\frac{-0.5\sqrt{c^2+1200ck+400c+4 \times 10^4 k^2 - 8 \times 10^4 k + 4 \times 10^4} + ck + 0.5c - 100k + 10}{k} < m <$ $\frac{c(-0.5\sqrt{c(800-317600k)+c^2+k(2.57282 \times 10^{10} k + 3.8368 \times 10^8) + 16 \times 10^4} + c(k+0.5) + 79800k + 200}{(c+160000)k}$
$c > 0, k < 0$ & $m < 0$	$m \leq -200, \frac{c(-c+m+200)+200m}{c^2+c(-2m-200)+m(m+200)} < k < \frac{c}{m-c}$
$c < 0, k < 0$ & $m < 0$	$c < m, \frac{c(-c+m+200)+200m}{c^2+c(-2m-200)+m(m+200)} < k < \frac{c}{m-c}$
$c < 0, k < 0$ & $m > 0$	$\frac{c}{c-m} + k > 0, k < \frac{c(c(-c+m+400)+400m}{c^3+c^2(-2m-400)+c(m-159600)m+1600m^2}$, $0.5m + 100.125 <$ $c + 0.000625782\sqrt{m(638401m + 7.6704 \times 10^8) + 2.56 \times 10^{10}}$
$c > 0, k < 0$ & $m > 0$	$0 < m < 79800, \frac{c}{c-m} + k > 0, c > 0.5m +$ $0.000625782\sqrt{m(638401m + 7.6704 \times 10^8) + 2.56 \times 10^{10}} + 100.125$, $k < \frac{c(-c+m+200)+200m}{c^2+c(-2m-200)+m(m+200)}$
$c < 0, k > 0$ & $m < 0$	$m > -1168, m < c$, $\frac{c(c(-c+m+400)+400m}{c^3+c^2(-2m-400)+c(m-159600)m+1600m^2} < k < \frac{-c}{c-m}$

In the Table 5.6-Table 5.8, it is found that there are all possible combinations of signs of c, k, and m with delay term $dt = 1, 0.5$ and 0.005 such that the fixed point $\left(\frac{-c}{c-m}, \frac{c(-k)-c+km}{k(c-m)^2}\right)$ is attracting except one which is remarked as follows.

Remark 5.1. There does not exists any real parameters $c < 0, k > 0$ and $m > 0$ such that the fixed point $\left(\frac{-c}{c-m}, \frac{c(-k)-c+km}{k(c-m)^2}\right)$ is attracting while the delay $dt = 1, 0.5$ and 0.005.

In Fig. 5.8.1-Fig. 5.8.3, a couple of examples of parameters c, k and m are provided such that the fixed point $\left(\frac{-c}{c-m}, \frac{c(-k)-c+km}{k(c-m)^2}\right)$ is attracting for different delay dt.

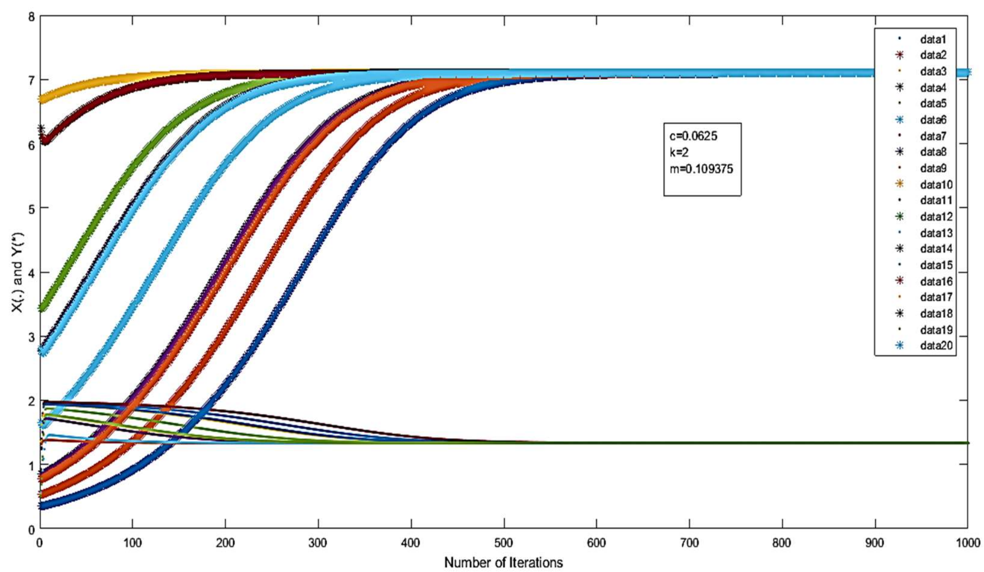


Fig 5.8.1. Example of parameters where the fixed point $\left(\frac{-c}{c-m}, \frac{c(-k)-c+km}{k(c-m)^2}\right)$ is attracting for $dt = 0.5$.

It is observed that for ten initial values taken from the neighborhood of the fixed point, trajectories are converging to $(1.33333, 7.11111)$.

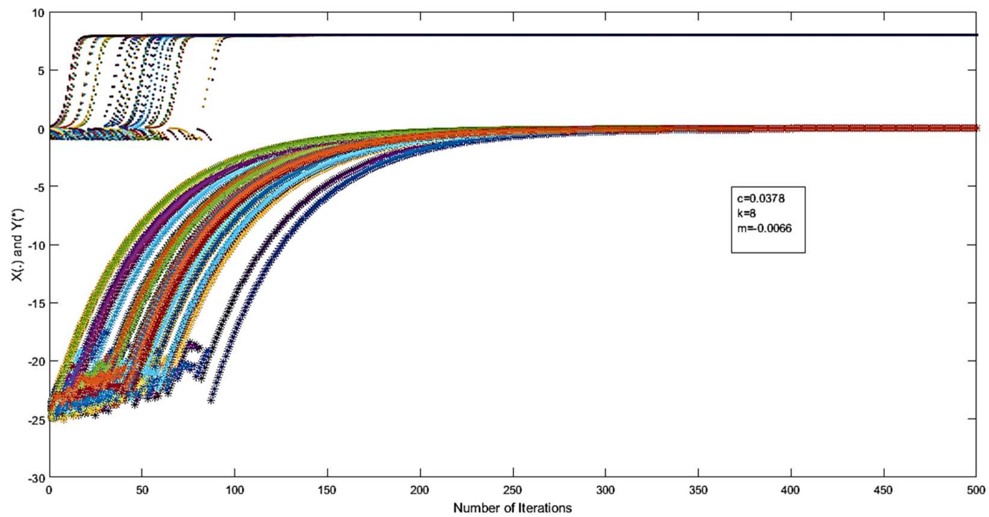


Fig 5.8.2. Example of parameters where the fixed point $\left(\frac{-c}{c-m}, \frac{c(-k)-c+km}{k(c-m)^2}\right)$ is attracting for $dt = 0.5$.

It is observed that for ten initial values taken from the neighborhood of the fixed point, the trajectories are attracting to the fixed point (8,0) which is not an expected outcome. The basin of attraction of the fixed point (8,0) is too big to attract the trajectories from even the neighborhood of the other fixed point.

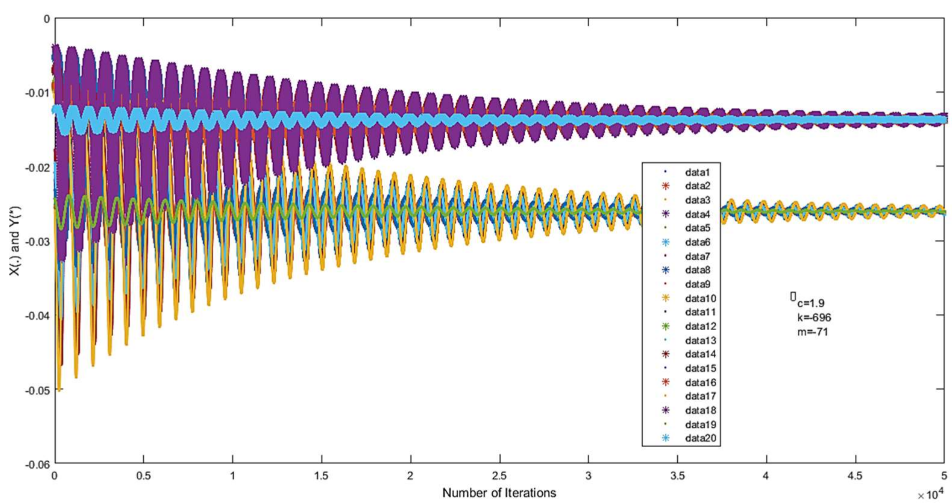


Fig 5.8.3. Example of parameters where the fixed point $\left(\frac{-c}{c-m}, \frac{c(-k)-c+km}{k(c-m)^2}\right)$ is attracting for $dt = 0.005$.

It was observed that for ten initial values taken from neighborhood of the fixed points, trajectories are converging to $(0.0481863, 0.692755)$.

We have conditions with different choices of sign of the parameters c, k and m in describing the local stability (repelling) conditions of the fixed point $\left(\frac{-c}{c-m}, \frac{c(-k)-c+km}{k(c-m)^2}\right)$.

Table 5.9. Local Asymptotic Stability (repelling) conditions for delay $dt = 1$.

Parameters with Sign	Conditions for repelling
$c > 0, k > 0$ & $m > 0$	$c < m, \frac{c(c(-c+m+2)+2m}{(c-m)((c-2)c-(c+4)m)} < k < \frac{c}{m-c}$
$c > 0, k < 0$ & $m < 0$	$m + 10 > 4\sqrt{6}, 2c + \sqrt{m(m+20)+4} > m + 2,$ $m + \sqrt{m(m+20)+4} + 2 > 2c, k < \frac{c(c(-c+m+2)+2m}{(c-m)((c-2)c-(c+4)m)}$
$c > 0, k < 0$ & $m > 0$	$\frac{c}{c-m} + k < 0, m < c, m + \sqrt{m(m+20)+4} + 2 \geq 2c$

Table 5.10. Local Asymptotic Stability (repelling) conditions for delay $dt = 0.5$.

Parameters with Sign	Conditions for repelling
$c > 0, k > 0$ & $m > 0$	$c < m, \frac{c(c(-c+m+4)+4m}{c^3+c^2(-2m-4)+c(m-12)m+16m^2} < k < \frac{c}{m-c}$
$c > 0, k < 0$ & $m < 0$	$-0.222912 < m, 0.5m - 0.5\sqrt{m(m+72)+16} + 2 < c <$ $0.5m + 0.5\sqrt{m(m+72)+16} + 2,$ $k < \frac{c(c(-c+m+4)+4m}{c^3+c^2(-2m-4)+c(m-12)m+16m^2}$
$c > 0, k < 0$ & $m > 0$	$\frac{c}{c-m} + k < 0, m < c, c \leq 0.5m + 0.5\sqrt{m(m+72)+16} + 2.$

Table 5.11. Local Asymptotic Stability (repelling) conditions for delay $dt = 0.005$.

Parameters with Sign	Conditions for repelling
$c > 0, k > 0$ & $m > 0$	$c < m, \frac{c(c(-c+m+4)+4m}{c^3+c^2(-2m-4)+c(m-12)m+16m^2} < k < \frac{c}{m-c}$
$c > 0, k < 0$ & $m < 0$	$-0.222912 < m, 0.5m - 0.5\sqrt{m(m+72)+16} + 2 < c <$ $0.5m + 0.5\sqrt{m(m+72)+16} + 2,$ $k < \frac{c(c(-c+m+4)+4m}{c^3+c^2(-2m-4)+c(m-12)m+16m^2}$
$c > 0, k < 0$ & $m > 0$	$\frac{c}{c-m} + k < 0, m < c, c \leq 0.5m + 0.5\sqrt{m(m+72)+16} + 2.$

In the Tables – 5.9,5.10 & 5.11, it is found that there are some sign combinations of c, k, and m with delay term $dt = 1, 0.5$ and 0.005 such that the fixed point $\left(\frac{-c}{c-m}, \frac{c(-k)-c+km}{k(c-m)^2}\right)$ is repelling except the ones which are remarked as follows.

Remark 5.2. There does not exists any real numbers satisfying
$\{(c > 0, k > 0, m < 0)\& (c < 0, k > 0, m < 0)\& (c < 0, k < 0, m > 0)\& (c < 0, k > 0, m > 0)\& (c < 0, k > 0, m < 0)\}$
such that the fixed point $\left(\frac{-c}{c-m}, \frac{c(-k)-c+km}{k(c-m)^2}\right)$ is repelling while the delay $dt = 1, 0.5$, and 0.05.

In Fig. 5.9.1 – Fig. 5.9.3, a couple of examples of parameters c, k and m were provided such that the fixed point $\left(\frac{-c}{c-m}, \frac{c(-k)-c+km}{k(c-m)^2}\right)$ is repelling.

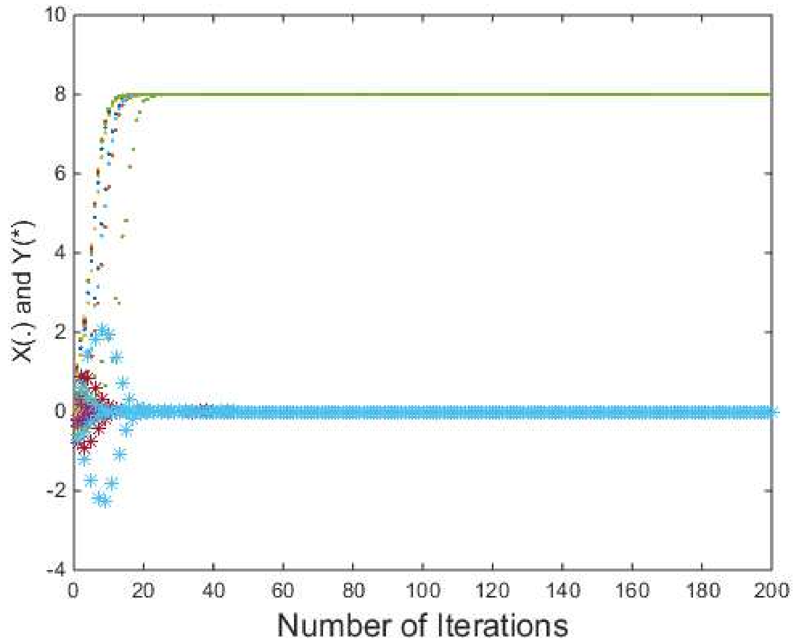


Fig 5.9.1. Example of parameters where the fixed point $\left(\frac{-c}{c-m}, \frac{c(-k)-c+km}{k(c-m)^2}\right)$ is repelling for $c = 5, k = 8, m = 2$ & $dt = 0.5$.

It is observed that for ten initial values taken from neighborhood of the fixed point, trajectories are repelling but attracting to the other fixed point (8,0).

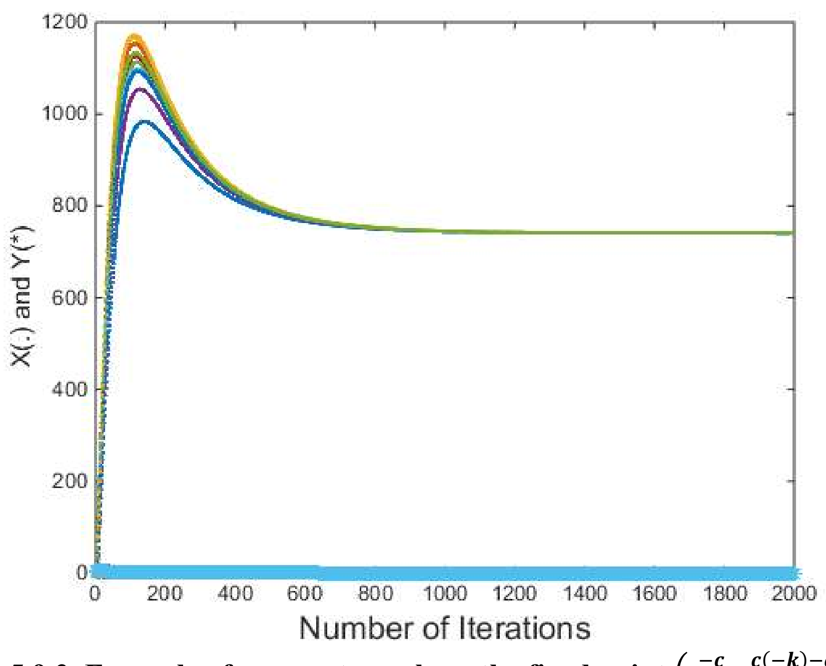


Fig 5.9.2. Example of parameters where the fixed point $\left(\frac{-c}{c-m}, \frac{c(-k)-c+km}{k(c-m)^2}\right)$ is repelling for $c = -867, k = 741, m = -872$ & $dt = 0.005$.

It is observed that for ten initial values taken from neighborhood of the fixed point, trajectories are repelling from the fixed point but attracting to the other fixed point $(741,0)$.

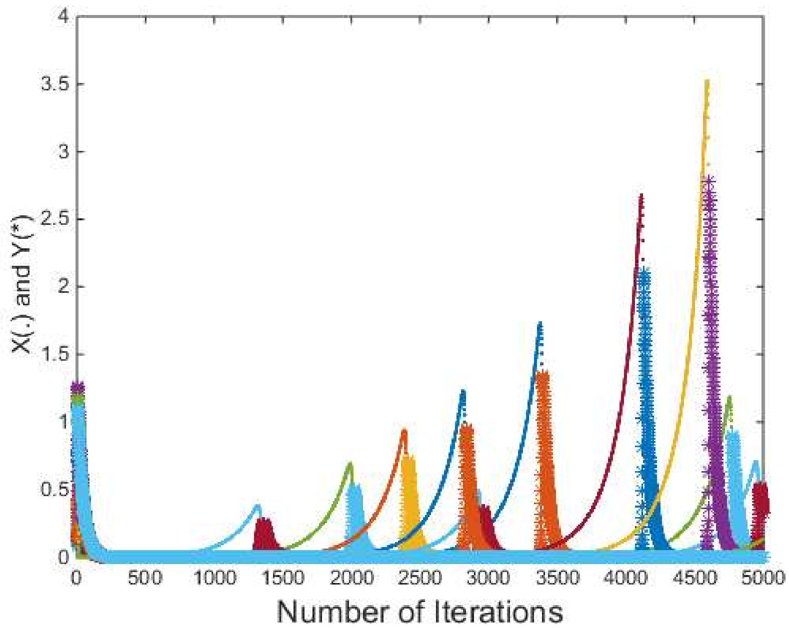


Fig 5.9.3. Example of parameters where the fixed point $\left(\frac{-c}{c-m}, \frac{c(-k)-c+km}{k(c-m)^2}\right)$ is repelling for $c = 5, k = -8, m = 92$ & $dt = 0.005$.

So far, the study explored the three-fixed point's local asymptotic stability of the Rosenzweig-Macarthur model eqs (5.3-5.4). In the following section, the higher order periodic solutions (limit cycles) of the same model were explored.

5.2.2 Limit Cycle Solutions of Eqs. (5.3-5.4)

Computationally, determined a set of parameters for attracting limit cycles [18], [23], [24]. The delay $dt = 0.005$ was fixed and number of iterations observed is 50000. The initial values are taken from neighborhood of all three fixed points (trajectories shown only for neighborhood of $(0,0)$). The parameter c is chosen as geometric mean, arithmetic mean and harmonic mean respectively of the other two parameters k and m as shown in Fig. 5.10.1-Fig. 5.10.12.

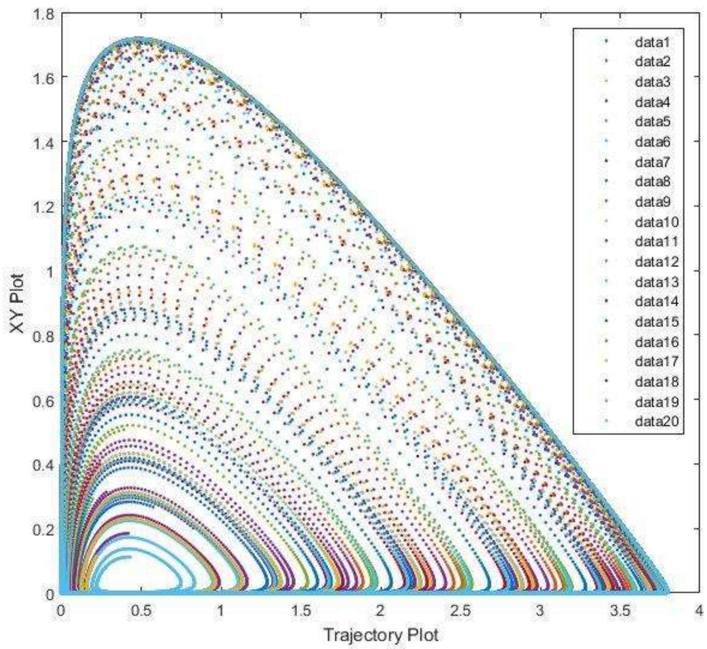


Fig 5.10.1. limit cycle for the parameters $k = 4, m = 46$ and $c = GM(k, m)$.

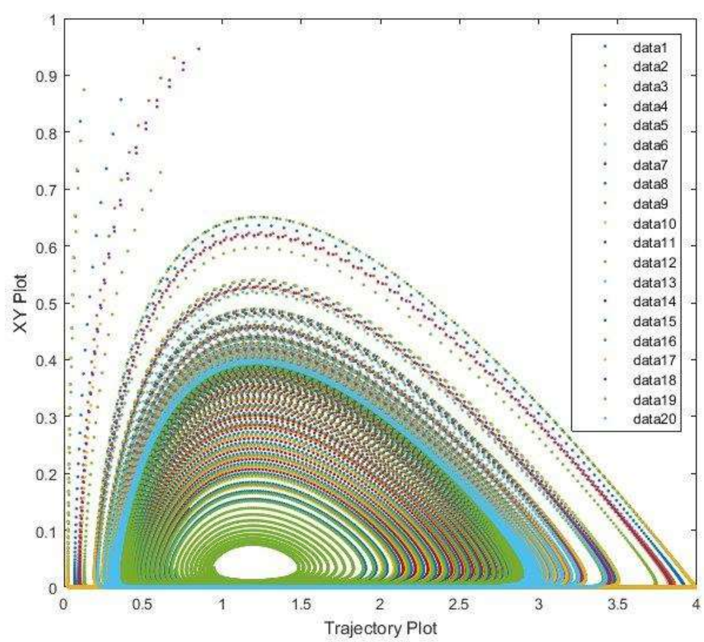


Fig 5.10.2. limit cycle for the parameters $k = 4, m = 46$ and $c = AM(k, m)$.

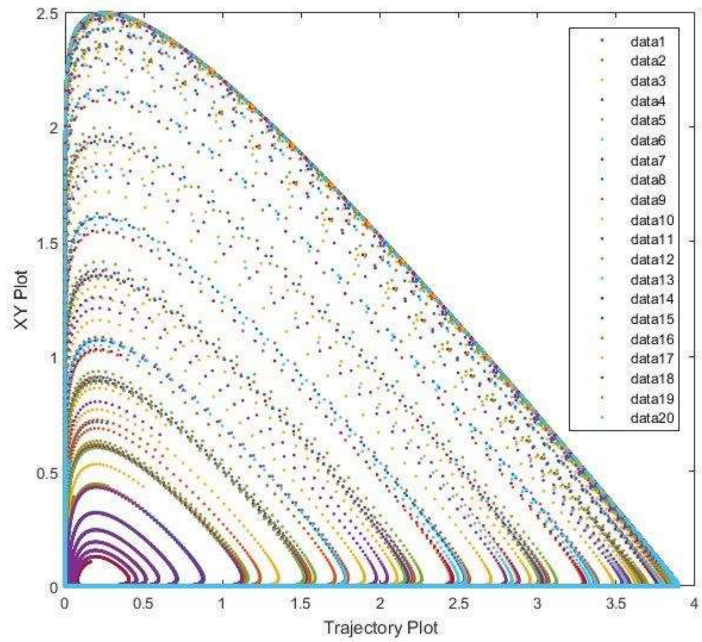


Fig 5.10.3. limit cycle for the parameters $k = 4, m = 46$ and $c = HM(k, m)$.

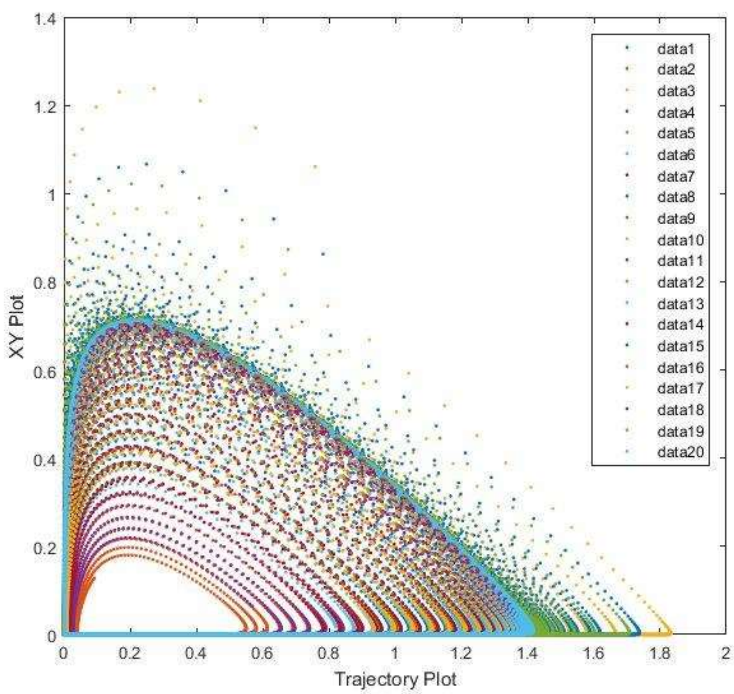


Fig 5.10.4. limit cycle for the parameters $k = 2, m = 80$ and $c = GM(k, m)$.

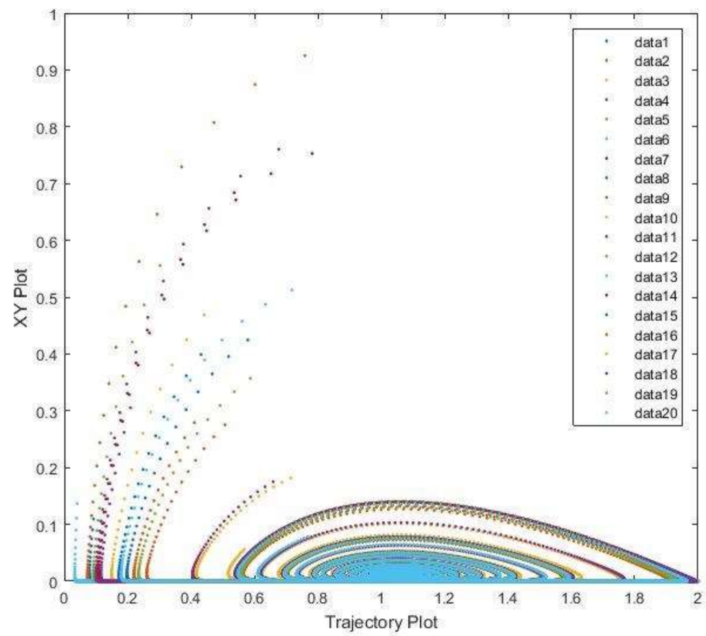


Fig 5.10.5. limit cycle for the parameters $k = 2, m = 80$ and $c = AM(k, m)$.

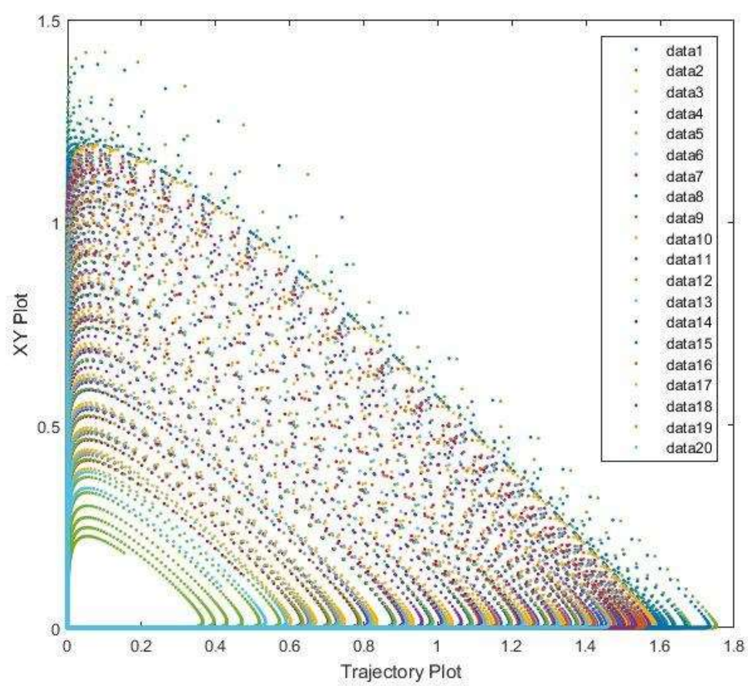


Fig 5.10.6. limit cycle for the parameters $k = 2, m = 80$ and $c = HM(k, m)$.

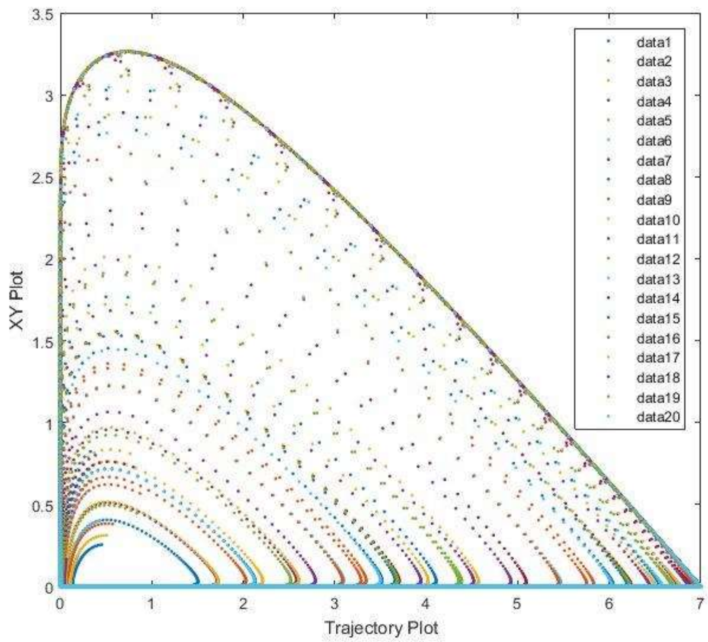


Fig 5.10.7. limit cycle for the parameters $k = 7, m = 65$ and $c = GM(k, m)$.

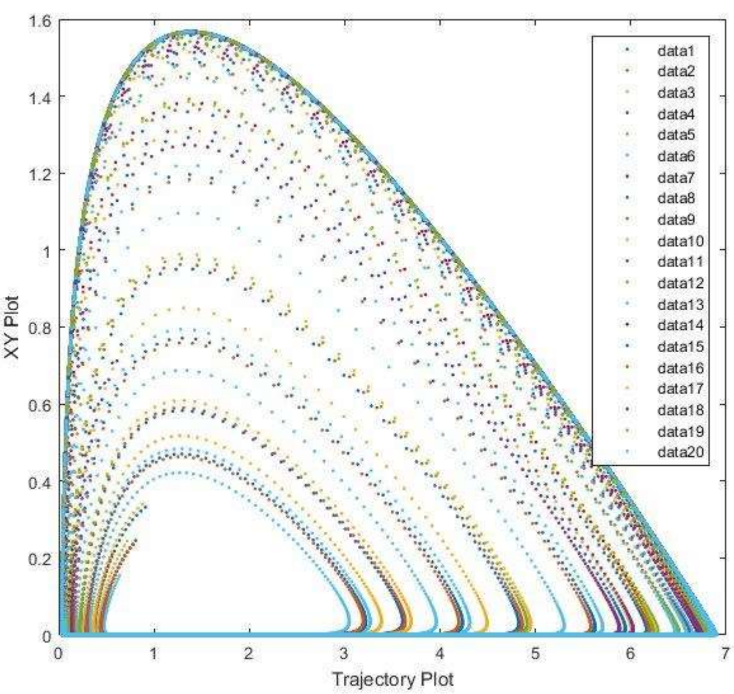


Fig 5.10.8. limit cycle for the parameters $k = 7, m = 65$ and $c = AM(k, m)$.

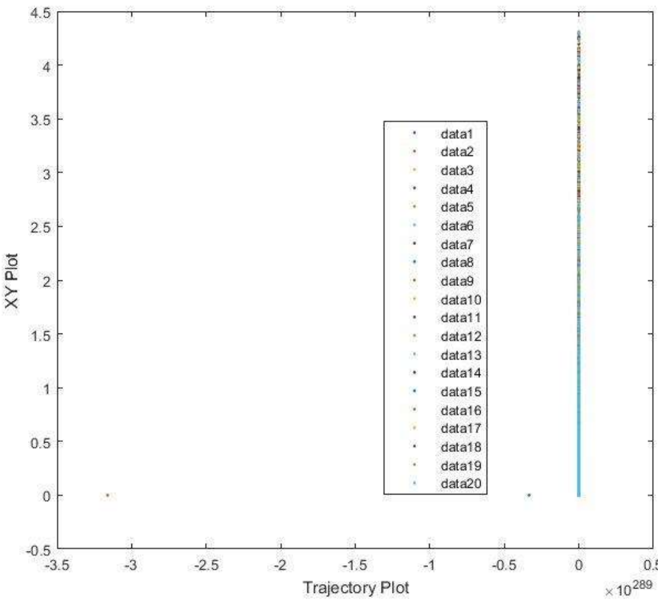


Fig 5.10.9. limit cycle for the parameters $k = 7, m = 65$ and $c = HM(k, m)$.

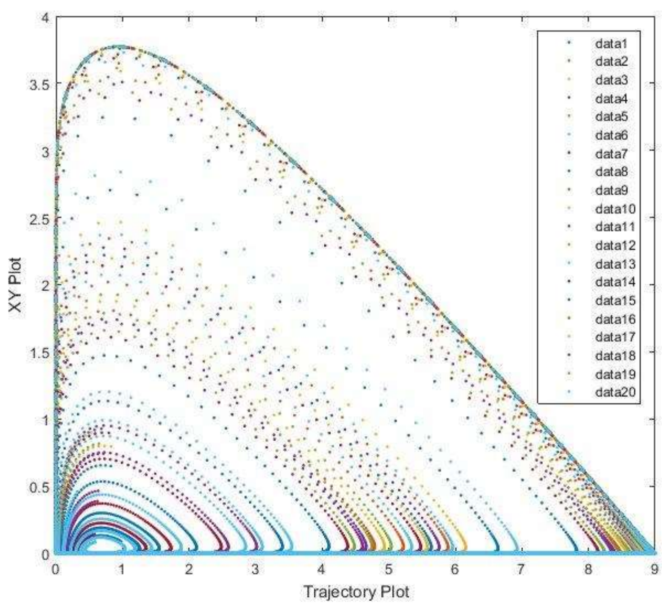


Fig 5.10.10. limit cycle for the parameters $k = 9, m = 56$ and $c = GM(k, m)$.

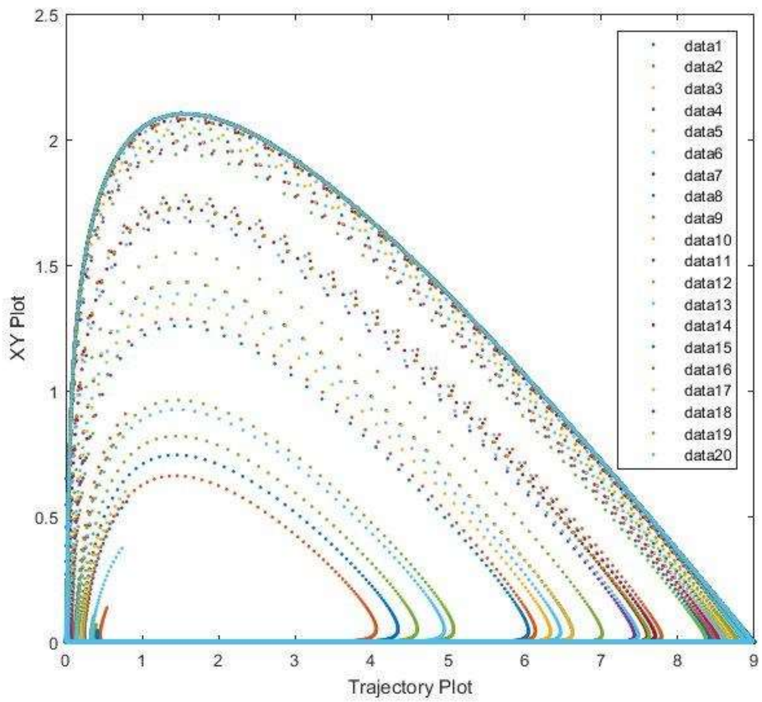


Fig 5.10.11. limit cycle for the parameters $k = 9, m = 56$ and $c = AM(k, m)$.

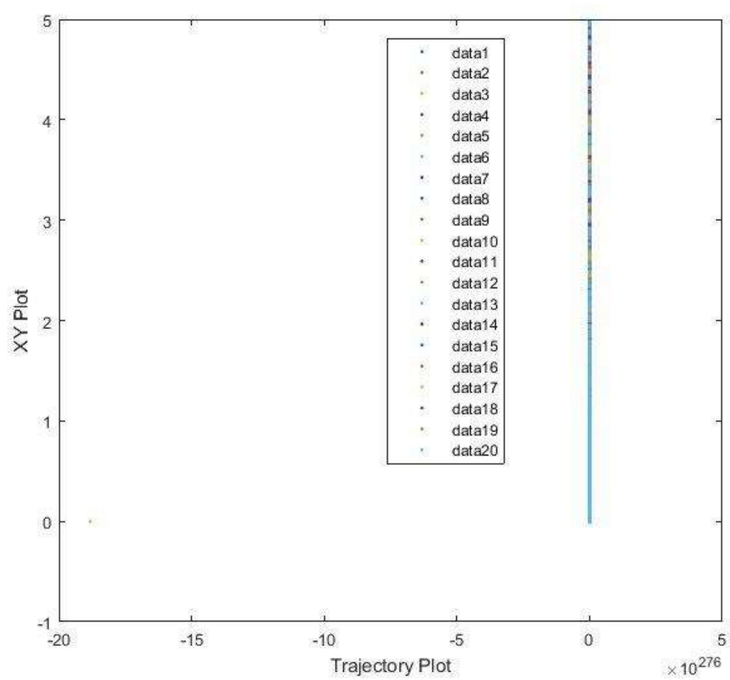


Fig 5.10.12. limit cycle for the parameters $k = 9, m = 56$ and $c = HM(k, m)$.

It is observed that, for the four sets of (k, m) values in Fig. 5.10.1-Fig. 5.10.12, the three fixed points of the models (5.3-5.4) are all repelling and consequently the limit cycles are attracting. Based on ample amount of simulations of similar instances, the following remark, observation and a counter example has been marked.

Remark 5.3. For delay $dt = 0.005$, there exists parameters $(c = AM(k, m), k, m)$ such that the limit cycle exist if and only if same happens true for $(c = GM(k, m), k, m)$.

Also observed that there does not exist any limit cycles for negative parameters $c, k,$ & m.

Example 5.1. There are parameters (c, k, m) such that the limit cycle exists where $c = AM(k, m)$ and $c = GM(k, m)$ but not for $c = HM(k, m)$ as stated in the 3rd and 4th row examples in Fig. 5.10.1 – Fig. 5.10.12.

5.3 Computational dynamics of Equations (5.5-5.6)

5.3.1 Local Stability Analysis

Here, the local asymptotic stability of the fixed points of the variant of the Rosenzweig-Macarthur model Eqs.(5.5-5.6) is discussed. The fixed points are $(k, 0), (0,0)$,

$\left(\frac{1}{2}\left(-\sqrt{k}\sqrt{4c + km^2 - 2km + k} + k(-m) + k\right), \frac{-\sqrt{k}m\sqrt{4c+km^2-2km+k}-2c-km^2+km}{2c}\right)$ and $\left(\frac{1}{2}\left(-\sqrt{k}\sqrt{4c + km^2 - 2km + k} + k(-m) + k\right), \frac{\sqrt{k}m\sqrt{4c+km^2-2km+k}-2c-km^2+km}{2c}\right)$.

For these four fixed points, the local asymptotic stability analysis was discussed by making the system of equations (5.5-5.6) linearized about the fixed points.

Local Stability Analysis of $(\boldsymbol{k}, \mathbf{0})$

The linearized system $X_{t+1} = JX_t$ (where $X_t = [x_t, y_t]^T$ and J is the Jacobian) is obtained by linearizing the model equations (5.5-5.6) about the fixed point $(k, 0)$. The Jacobian about the fixed point $(k, 0)$ is

$$J_{(k,0)} = \begin{pmatrix} 1 - dt & -dtkm \\ 0 & dt(km - c) + 1 \end{pmatrix}$$

The eigenvalues are $1 - dt$ and $dt(km - c) + 1$. In the following tables Table 5.12-Table 5.14, the attracting, repelling and saddle conditions are given for the fixed point $(k, 0)$ respectively.

Table 5.12. Eigenvalues and corresponding conditions for attracting of $(k, 0)$ for different delays.

Delay Term (dt)	Eigenvalues	Conditions
1	$0 \; -c + km + 1$	$m > 0, \frac{c-2}{m} < k < \frac{c}{m}$
0.5	0.5 & $0.5(km - c) + 1$	$m > 0, \frac{c-4}{m} < k < \frac{c}{m}$
0.005	0.995 & $0.005(km - c) + 1$	$m > 0, \frac{c-400}{m} < k < \frac{c}{m}$

Table 5.13. Eigenvalues and corresponding conditions for repelling of $(k, 0)$ for different delays.

Delay Term (dt)	Eigenvalues	Conditions
1	0 & $1 - c + km$	$m < 0, k < \frac{c}{m}$
0.5	0.5 & $1 + 0.5(-c + km)$	$m < 0, k < \frac{c}{m}$
0.005	0.995 & $1 + 0.005(-c + km)$	$m < 0, k < \frac{c}{m}$

Table 5.14. Eigenvalues and corresponding conditions for saddle of $(k, 0)$ for different delays.

Delay Term (dt)	Eigenvalues	Conditions
1	0 & $1 - c + km$	$c = km$
0.5	0.5 & $1 + 0.5(-c + km)$	$c = km$
0.005	0.995 & $1 + 0.005(-c + km)$	$c = km$

Here the visualizations in three-dimensional subspaces $T_{(attracting,dt)}$, $T_{(repelling,dt)}$, and $T_{(saddle,dt)}$ of $\mathbb{R}^3$ for different delays dt, which are shown in Fig 5.11.1-Fig.5.11.3 for model Eqs. (5.5-5.6). In precise, $T_{(attracting,dt)}$ denotes the space of parameters (c, k, m) in $\mathbb{R}^3$ for which the fixed point $(k, 0)$ is attracting and similarly others follow.

Here a couple of examples of parameters (c, k, m) for which $(k, 0)$ is attracting are demonstrated as shown in Table 5.15.

Table 5.15. Examples of parameters where $(k, 0)$ is attracting for different delays.

Delay Term (dt)	**Parameters (c, k, m)**	**Nature**
1	$(38, -15, -2.43668)$	Attracting to $(-15,0)$
1	$(-113, 86, -1.314)$	Attracting to $(86,0)$
0.5	$(3.69841, -67, -0.0455083)$	Attracting to $(-67,0)$
0.5	$(-267, 84, -3.22479)$	Attracting to $(84,0)$
0.005	$(204, 6, -9.5)$	Attracting to $(6,0)$
0.005	$(-267, 84, -7.81818)$	Attracting to $(84,0)$

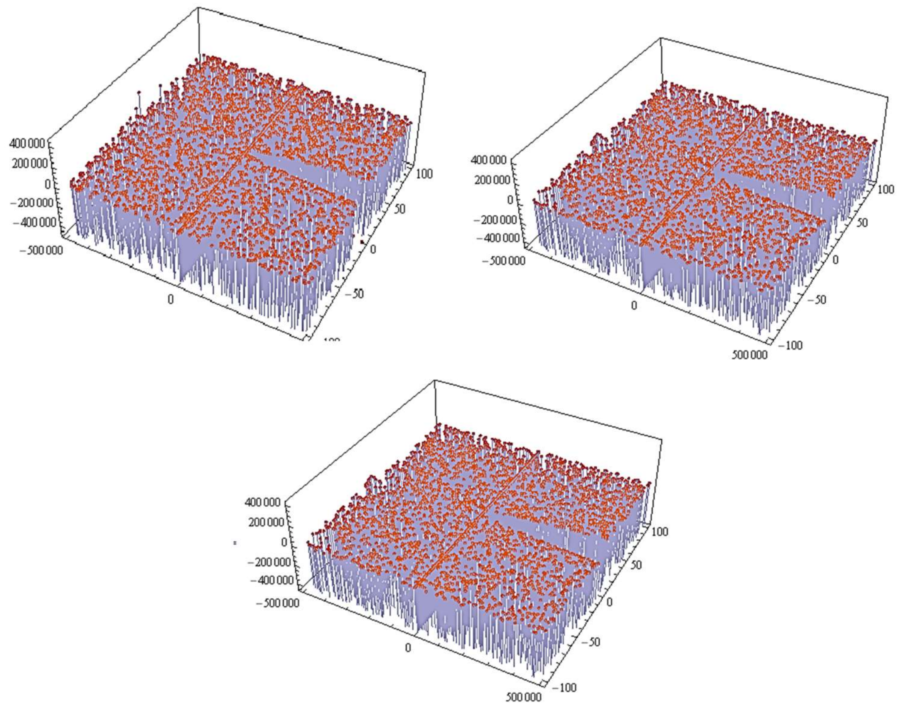


Fig 5.11.1. Top Left: $T_{(attracting,1)}$, Top Right: $T_{(attracting,0.5)}$,

Bottom: $T_{(attracting,0.005)}$.

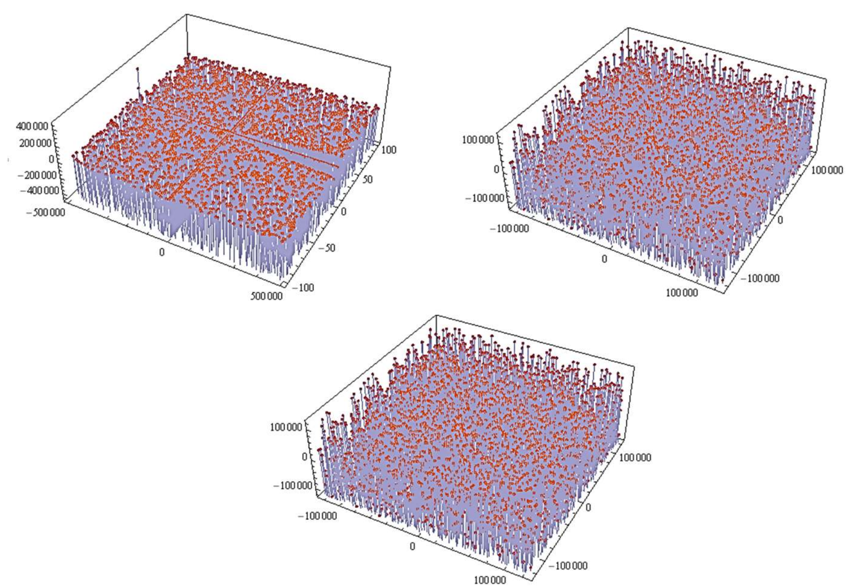


Fig 5.11.2. Top Left: $T_{(repelling,1)}$, Top Right: $T_{(repelling,0.5)}$,

Bottom: $T_{(repelling,0.005)}$.

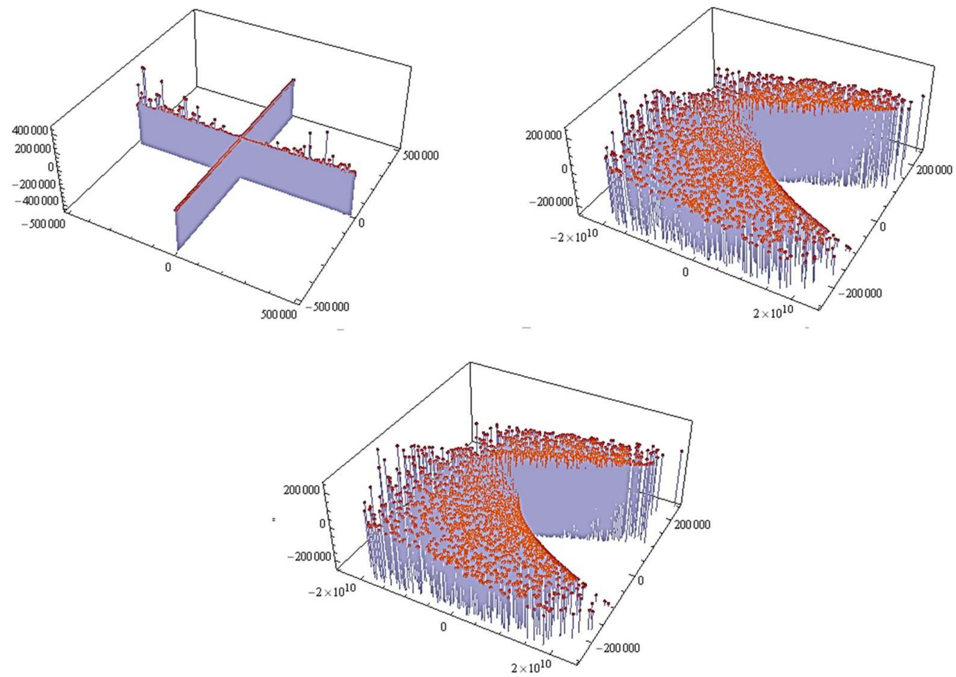


Fig 5.11.3. Top Left: $T_{(saddle,1)}$, Top Right: $T_{(saddle,0.5)}$,

Bottom: $T_{(saddle,0.005)}$.

The trajectory plot in the following Fig. 5.12.1-Fig. 5.12.2 is shown for each examples given in Table 5.15.

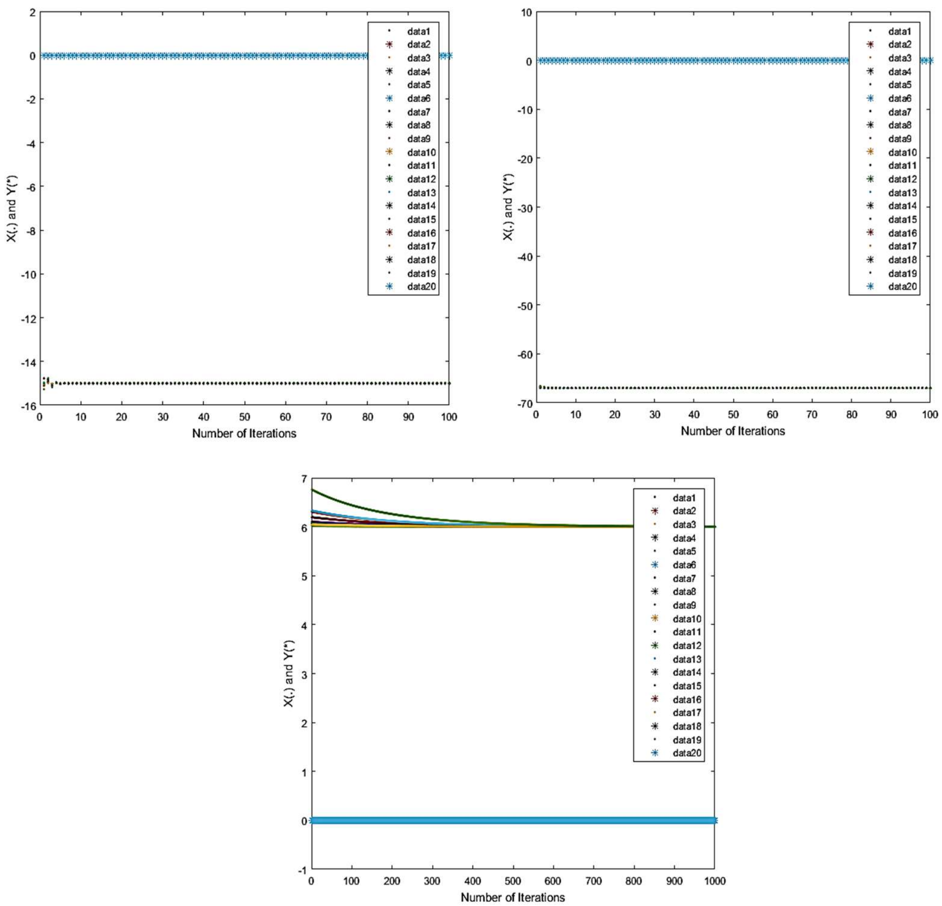


Fig. 5.12.1. Examples of trajectories where the fixed point $(k, 0)$ is attracting. Top Left: $dt = 1$, Top Right: $dt = 0.5$, Bottom: $dt = 0.005$.

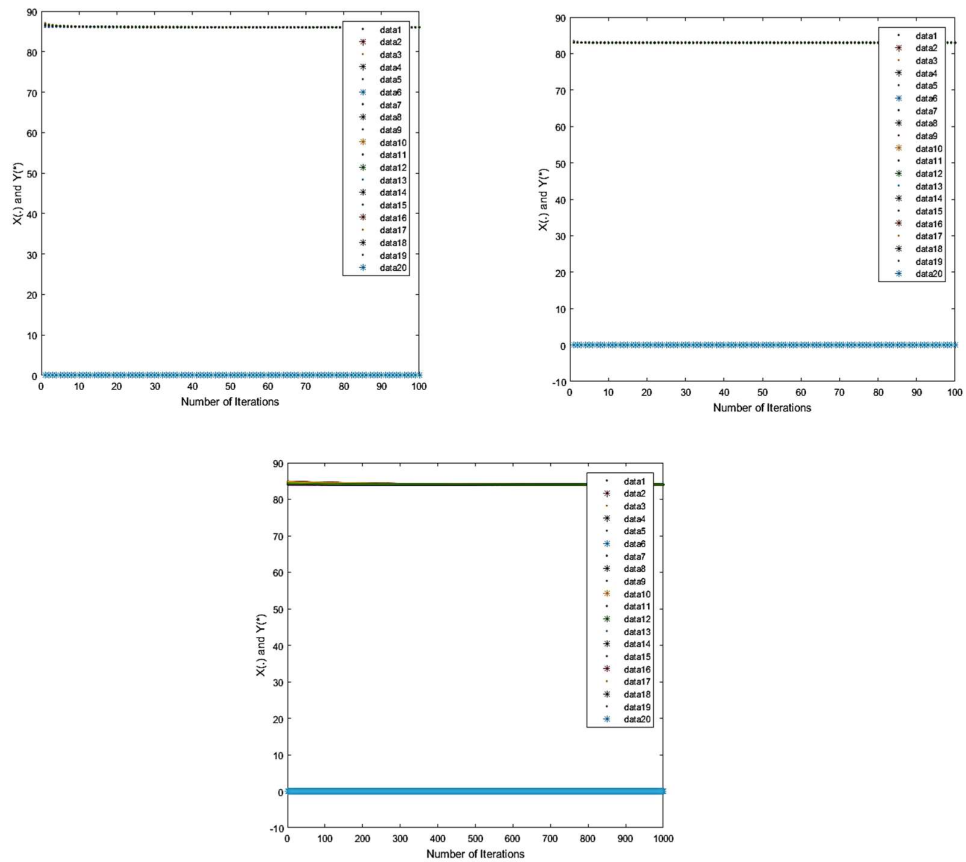


Fig. 5.12.2. Examples of trajectories where the fixed point $(\boldsymbol{k}, \mathbf{0})$ is attracting. Top Left: $\boldsymbol{dt} = \mathbf{1}$, Top Right: $\boldsymbol{dt} = \mathbf{0.5}$, Bottom: $\boldsymbol{dt} = \mathbf{0.005}$.

Now considered another example of parameters $c = 113.2, k = 103.5$ & $m = 62.4$ and delay $dt = 0.005$ where the fixed point $(k, 0)$ is repelling as shown in Fig 5.13. In the neighborhood of $(k, 0)$, for 10 different initial value sets, it is observed that the convergence of trajectories to the other fixed point $(1.8431, 0.0160)$ is taking place as shown in Fig 5.13.

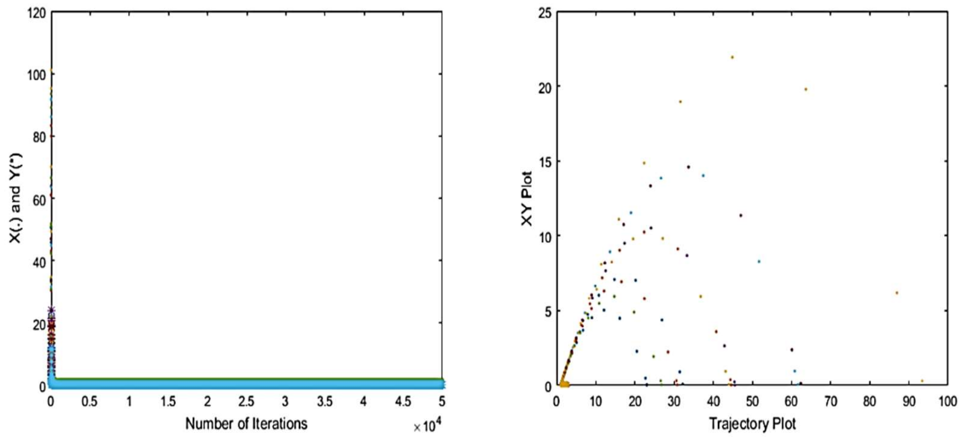


Fig. 5.13. Examples of trajectories where the fixed point $(\boldsymbol{k}, \mathbf{0})$ is repelling.

Local Stability Analysis of $(\mathbf{0}, \mathbf{0})$

The linearized system $X_{t+1} = JX_t$, where $X_t = [x_t, y_t]^T$ and J is the Jacobian is obtained by linearizing the model equations Eqn. (5.5-5.6) about the fixed point (0,0).

The Jacobian about (0,0) is given by

$$J_{(0,0)} = \begin{pmatrix} dt+1 & 0 \\ 0 & 1-cdt \end{pmatrix}$$

The eigenvalues are $dt + 1$ and $1 - cdt$. Therefore for any delay term $dt > 0$, there does not exists any parameter such that the absolute value of both the eigenvalues are less than one. Hence, the fixed point (0,0) can never be attracting as in the case of Eqs. (5.3-5.4).

Theorem 5.3. The fixed point (0,0) of the model Eqs (5.5-5.6) is repelling for $dt > 0$ and $cdt < 2$ holds.

Proof. Straight forward from Result 5.1

The point (0,0) exhibits saddle behavior for delay $dt = 0.005$ and $c = 0, k = 50$ and $m = 4$ for ten distinct initial value sets as depicted in Fig. 5.10. Here from a small neighborhood of the point (0,0), all the ten initial conditions were chosen.

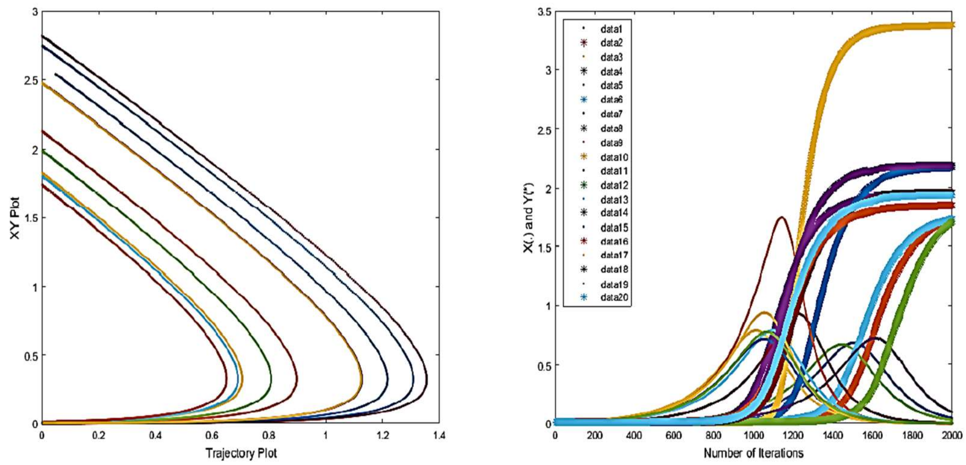


Fig 5.14. Example of trajectories where the fixed point $(\mathbf{0},\mathbf{0})$ is saddle.

Local Stability Analysis of the fixed point

$$\left(\frac{1}{2}\left(-\sqrt{k}\sqrt{4c+km^2-2km+k}+k(-m)+k\right),\frac{-\sqrt{k}m\sqrt{4c+km^2-2km+k}-2c-km^2+km}{2c}\right)$$

Before proceeding to local stability analysis, the interest is in parameters c, k *and* m such that the fixed point is real. Following the four subsets of $\mathbb{R}^3$ of parameters (c, k, m) are obtained such that the fixed point lies in the I, II, III and IV quadrants respectively as shown in Fig 5.15.

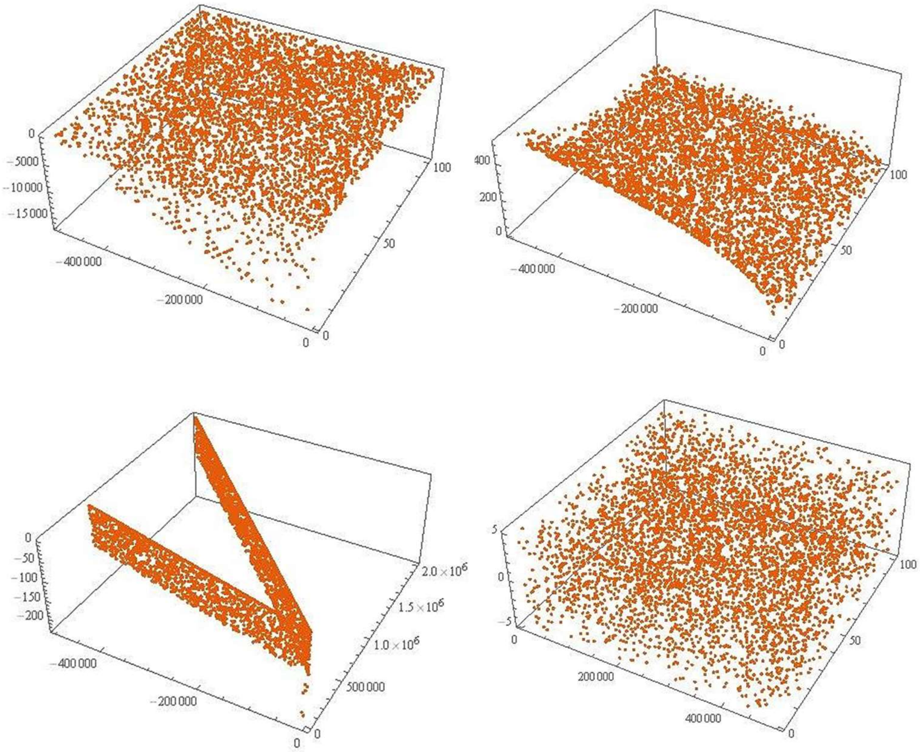


Fig 5.15. Subspace of parameters such that the fixed point lies in the first quadrant (Top Left), second quadrant (Top Right), third quadrant (Bottom Left) and fourth quadrant (Bottom Right).

The conditions for the fixed point to lie in I, II, III and IV quadrants respectively are shown below:

Result 5.2. The fixed point

$$\left(\frac{1}{2}\left(-\sqrt{k}\sqrt{4c+km^2-2km+k}+k(-m)+k\right), \frac{-\sqrt{k}m\sqrt{4c+km^2-2km+k}-2c-km^2+km}{2c}\right)$$

lies in the

- I quadrant if $c < 0, m - 1, \& \frac{-4c}{(m-1)^2} \leq k \leq \frac{c}{m}$
- II quadrant if $m > 1; c < 0 \ \& \ k \geq \frac{-4c}{(m-1)^2}$
- III quadrant if both $\{c, k\} > 0$
- IV quadrant if $c < 0, k > \frac{c}{m} \ \& \ m \leq -1$

The Jacobian about the fixed point is given by

$$J = \begin{pmatrix} \frac{1}{2}\left(dt\left(m+\frac{\sqrt{k(m-1)^2+4c}}{\sqrt{k}}-1\right)+2\right) & \frac{2c^2dt}{k(m-1)m+\sqrt{k}\sqrt{k(m-1)^2+4cm}} \\ \frac{1}{2}dt\left(m+\frac{\sqrt{k(m-1)^2+4c}}{\sqrt{k}}+1\right) & -\frac{cdt\left(m+\frac{\sqrt{k(m-1)^2+4c}}{\sqrt{k}}+1\right)-2m}{2m} \end{pmatrix}$$

Consider D and T to be respectively determinant and trace of the Jacobian J, the from the Result 5.1 the following theorem was proposed.

Theorem 5.4. The fixed point of the model Eqs (5.5-5.6)

$$\left(\frac{1}{2}\left(-\sqrt{k}\sqrt{4c+km^2-2km+k}+k(-m)\right.\right.$$
$$\left.\left.+k\right), \frac{-\sqrt{k}m\sqrt{4c+km^2-2km+k}-2c-km^2+km}{2c}\right)$$

is attracting if $|T| < 1 + D < 2$.

Based on computational simulation with delay $dt = 1$, following remarks are made.

Remark 5.4. There does not exist any parameters $c = k = m$ such that the above theorem holds.

Remark 5.5. If any one of the parameters is zero, then the above Theorem fails. While we consider the delay $dt = 0.005$, through simulation only one instance of the parameters $c = \frac{44}{5}, k = -\frac{53}{10}$, and $m = -\frac{17}{10}$ is found for which the fixed point

$$\left(\frac{1}{2}\left(-\sqrt{k}\sqrt{4c + km^2 - 2km + k} + k(-m) + k\right), \frac{-\sqrt{k}m\sqrt{4c + km^2 - 2km + k} - 2c - km^2 + km}{2c}\right)$$

is complex. Hence there does not exists any real parameters such that the fixed point (real) is attracting. Based on the computational evidence, the following conjecture is made.

Conjecture 5.1. There does not exist any real parameters (c, k, m) and delay $dt > 0$ such that the fixed point

$$\left(\frac{1}{2}\left(-\sqrt{k}\sqrt{4c + km^2 - 2km + k} + k(-m) + k\right), \frac{-\sqrt{k}m\sqrt{4c + km^2 - 2km + k} - 2c - km^2 + km}{2c}\right)$$

is attracting.

Here the subspace of parameters in $\mathbb{R}^3$ for different delay $dt = 0.5, 0.005$ and 0.0005 was obtained such that the fixed point is repelling which is shown in Fig. 5.16.

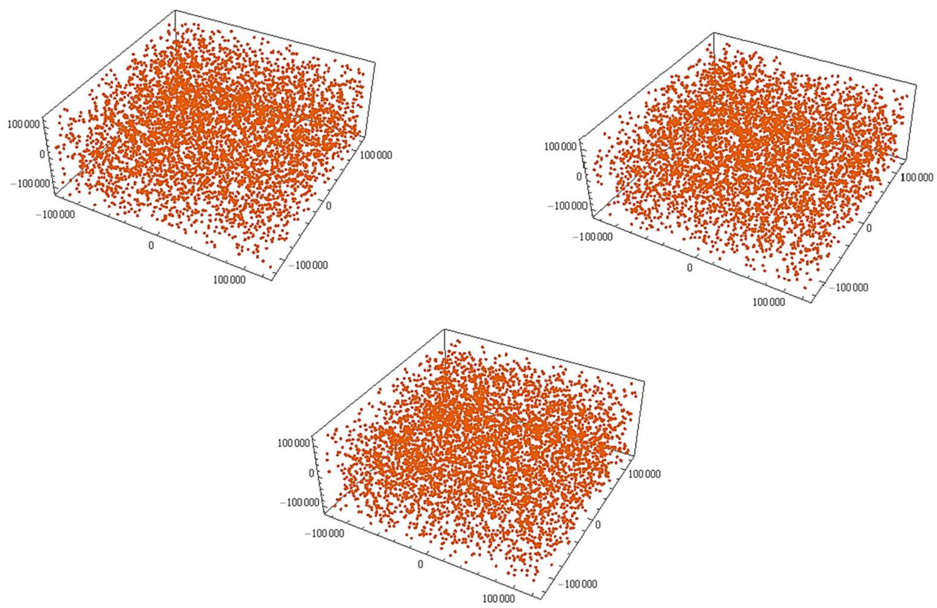


Fig 5.16. Subspace of parameters such that the fixed point is repelling for $dt = 0.5, 0.05$ and 0.0005 in Top Left, Top Right and Bottom of the figure respectively.

Local Stability Analysis of

$$\left(\frac{1}{2}\left(-\sqrt{k}\sqrt{4c+km^2-2km+k}+k(-m)+k\right),\frac{\sqrt{k}m\sqrt{4c+km^2-2km+k}-2c-km^2+km}{2c}\right).$$

Before proceeding to local stability analysis, the interest is in parameters c, k and m such that the fixed point is real. Following the four subsets of $\mathbb{R}^3$ of parameters (c, k, m) were obtained such that the fixed point lies in the I, II, and IV quadrants respectively as shown in Fig 5.17.

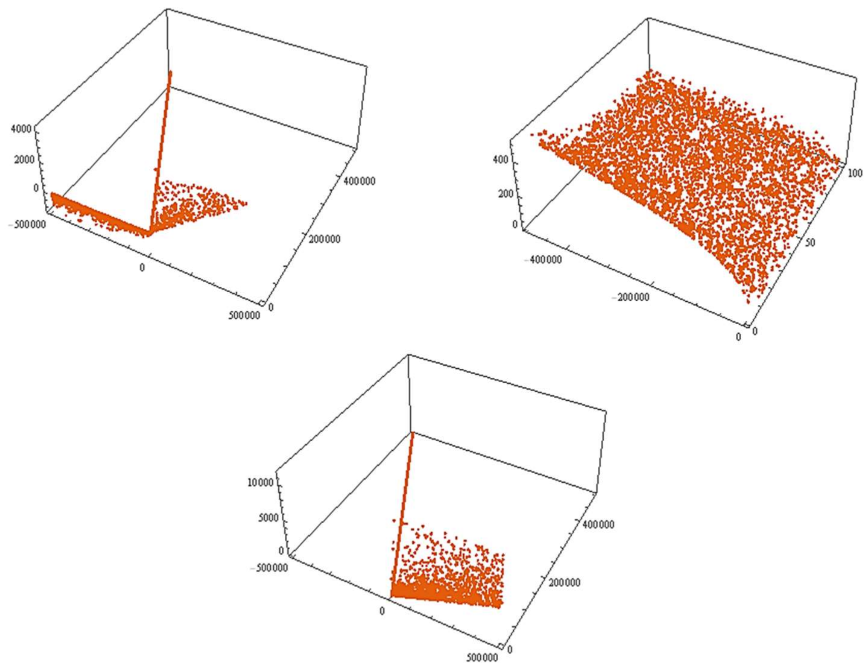


Fig. 5.17. Subspace of parameters such that the fixed point lies in the first quadrant (Top Left), second quadrant (Top Right), and fourth quadrant (Bottom).

The conditions for the fixed point to lie in the first, second, fourth quadrants respectively are shown below:

Result 5.3. The fixed point

$$\left(\frac{1}{2}\left(-\sqrt{k}\sqrt{4c+km^2-2km+k}+k(-m)+k\right),\frac{\sqrt{k}m\sqrt{4c+km^2-2km+k}-2c-km^2+km}{2c}\right)$$

lies in the

- I quadrant if $c < 0, m - 1, \& \ k \geq \frac{-4c}{(m-1)^2}$

- II quadrant if $m > 1; c < 0$ & $k \geq \frac{-4c}{(m-1)^2}$
- IV quadrant if $c > 0, k > 0$ & $m \leq 0$

Remark 5.6. There does not exist any real parameter c, k and m such that the fixed point lies in the III quadrant.

The Jacobian about the fixed point is given by

$$J = \begin{pmatrix} \frac{1}{2}\left(dt\left(m - \frac{\sqrt{k(m-1)^2+4c}}{\sqrt{k}} - 1\right) + 2\right) & \frac{2c^2dt}{k(m-1)m - \sqrt{k}\sqrt{k(m-1)^2+4cm}} \\ \frac{1}{2}dt\left(m - \frac{\sqrt{k(m-1)^2+4c}}{\sqrt{k}} + 1\right) & -\frac{cdt\left(m - \frac{\sqrt{k(m-1)^2+4c}}{\sqrt{k}} + 1\right) - 2m}{2m} \end{pmatrix}$$

Consider D and T to be respectively determinant and trace of the Jacobian J, the from the Result 5.1 we have the following theorem 5.5.

Theorem 5.5. The fixed point of the model Eqs (5.5-5.6)

$$\left(\frac{1}{2}\left(-\sqrt{k}\sqrt{4c + km^2 - 2km + k} + k(-m) + k\right), \frac{\sqrt{k}m\sqrt{4c + km^2 - 2km + k} - 2c - km^2 + km}{2c}\right)$$

is attracting if $|T| < 1 + D < 2$.

Based on computational simulation with delay $dt = 0.005$, following remarks are made.

Remark 5.7. The theorem 5.5 holds well if the parameters are all equal such that $1 < k < 201.99$.

Remark 5.8. If the parameter $c = 0$, then the Theorem 5.5 holds provided $k > 0$ and $-273.818 < m < 0.818061$.

Remark 5.9. There does not exist any non-zero parameter c and $m = k = 0$ such that the Theorem 5.5 holds.

Here presented a set of examples of parameters such that the Theorem 5.5 holds well.

Example 5.2. Consider $c = 3, k = 5$, and $m = 1$. From the neighborhood of the fixed point $(3.87298, 0.290994)$, 10 different initial values were chosen and it is noticed that the fixed point $(3.87298, 0.290994)$ which lies in the first quadrant is attracting which is depicted in the following Fig 5.18.

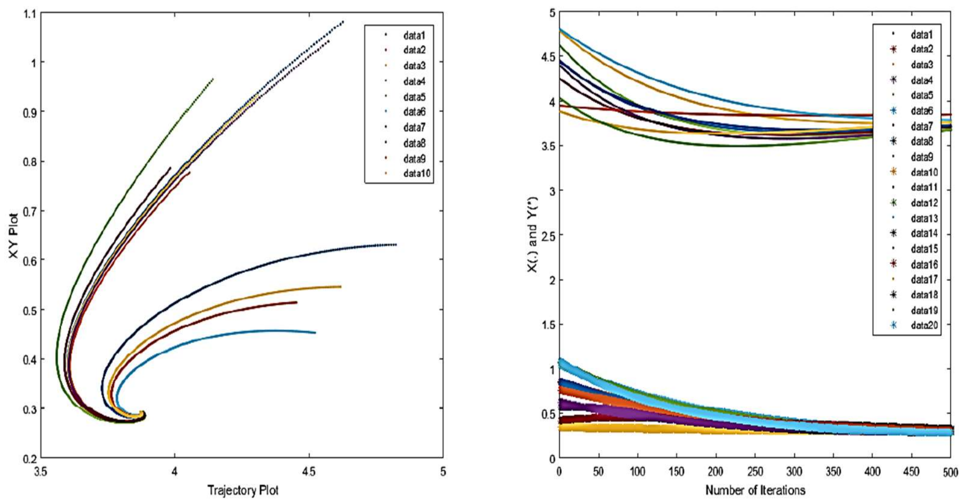


Fig 5.18. Plot of attracting trajectories.

Example 5.3. Consider $k = -65, m = 92$, and $c = AM(k, m)$. From the neighborhood of the point $(0.148, 0.011)$, 10 different initial values were chosen and it is noticed that the fixed point $(0.148, 0.011)$ which lies in the first quadrant is attracting which is depicted in the following Fig 5.19.

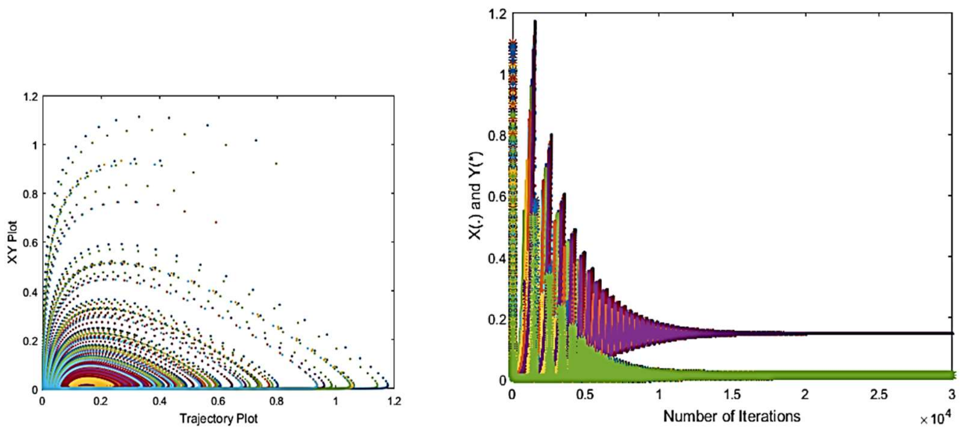


Fig 5.19. Plot of attracting trajectories

Here the subspace of parameters in $\mathbb{R}^3$ for different delay $dt = 0.5, 0.005$ and 0.0005 was obtained such that the fixed point is repelling which is shown in Fig 5. 20.

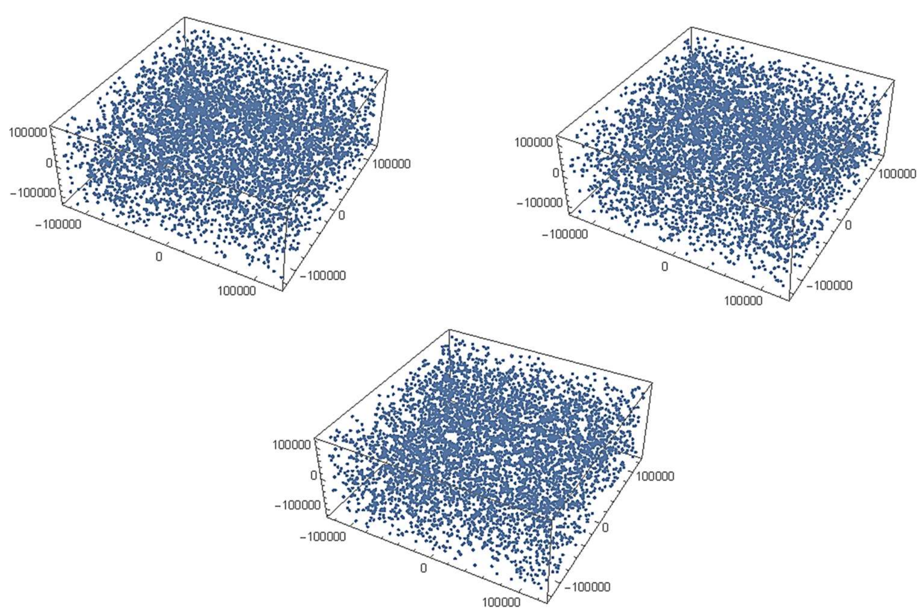


Figure 5.20. Subspace of parameters such that the fixed point is repelling for $dt = 0.5, 0.05$ and 0.0005 in Top Left, Top Right and Bottom of the figure respectively.

If we consider the delay $dt = 0.005$ with parameters $c = 75, k = 1$ and $m = 10$, the trajectories are repelling (attracting to the other fixed point $(k, 0)$) from $(5.25961, -0.298719)$ for any initial conditions taken from neighborhood as shown in Fig 5.21.

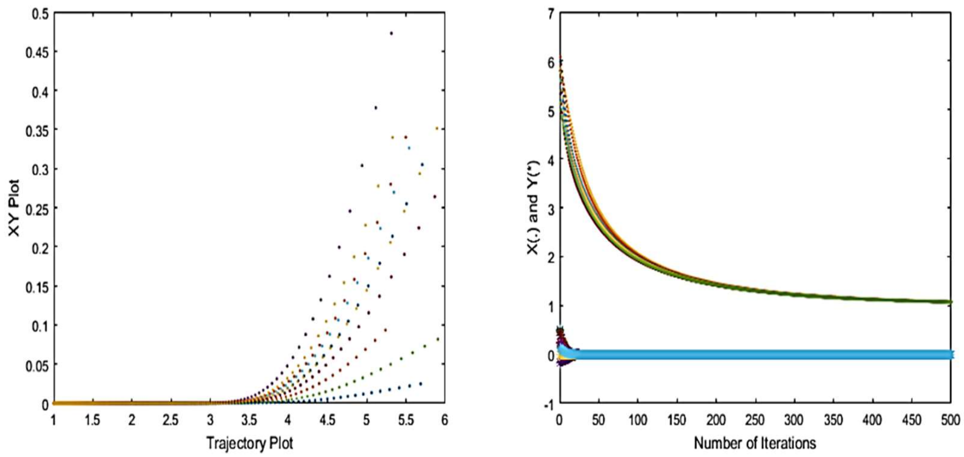


Fig 5.21. Plot of repelling trajectories.

Consider another example of parameters $c = -42, k = -47,$ and $m = -37$ with the delay $dt = 0.005$, where the trajectories are repelling (making limit cycle) from $(1.10458, -0.269176)$ for any initial conditions taken from neighborhood as shown in Fig 5.22. It is noted that, the fixed point corresponding to these parameters is no more real, it is complex number through.

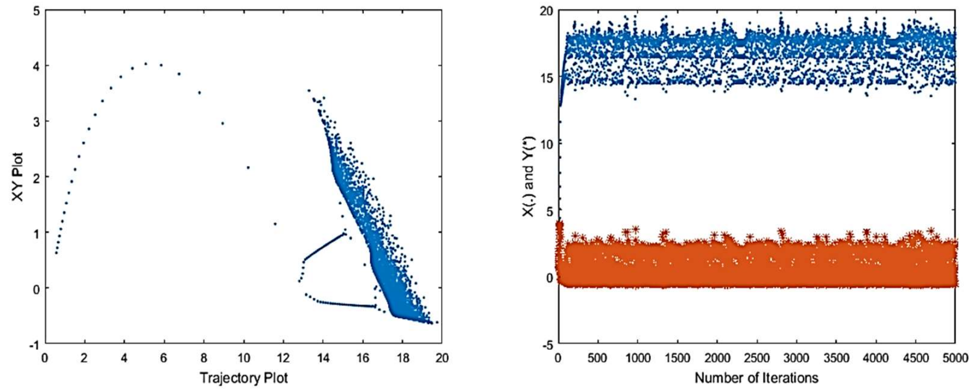


Fig 5.22. Plot of repelling trajectories.

5.4 Limit cycle solutions of Eqs. (5.5-5.6)

Here a set of parameters were determined computationally for attracting limit cycles of the model Eqs (5.5-5.6).As previous one, here the delay $dt = 0.005$ was fixed with different number of iterations and initial values are taken from neighborhood of all three fixed points (trajectories are shown only for **neighborhood of $(\mathbf{0}, \mathbf{0})$) . Considered the parameter c as arithmetic mean of other two parameters k and m as shown in the figures Fig. 5.23.1-Fig. 5.23.3.**

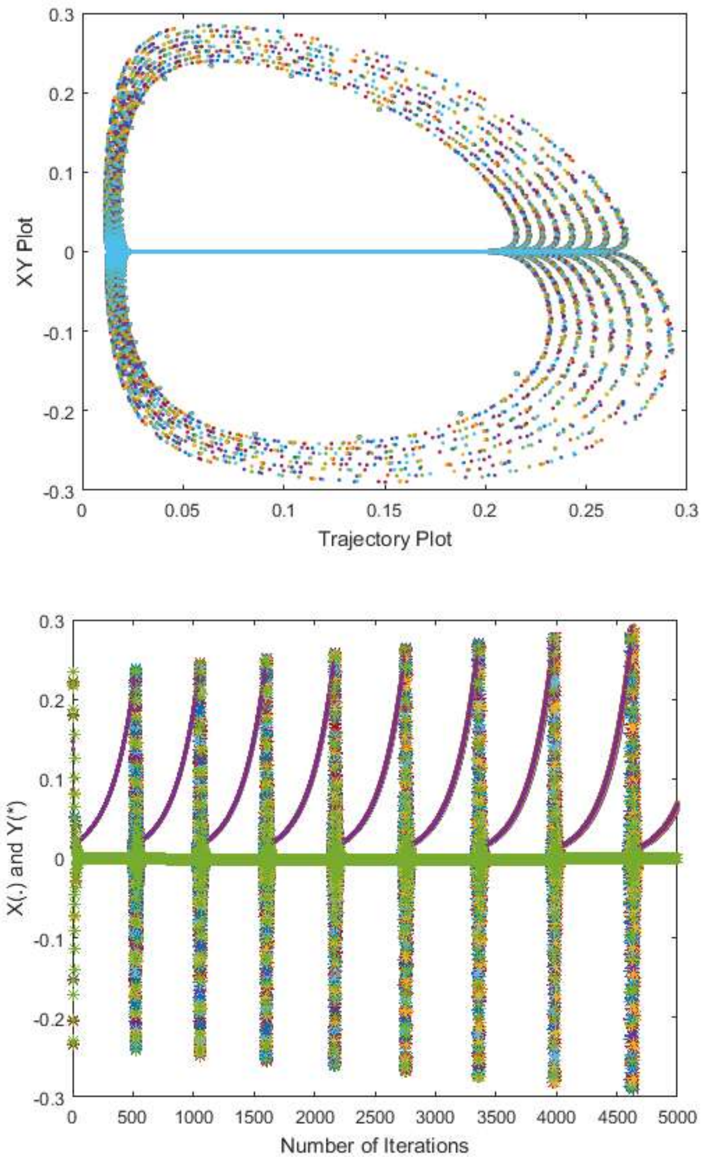


Fig 5.23.1. Limit Cycle (above) and trajectories (below) for the parameters $\boldsymbol{k = -388, m = -355}$ & $\boldsymbol{c = GM(k, m)}$.

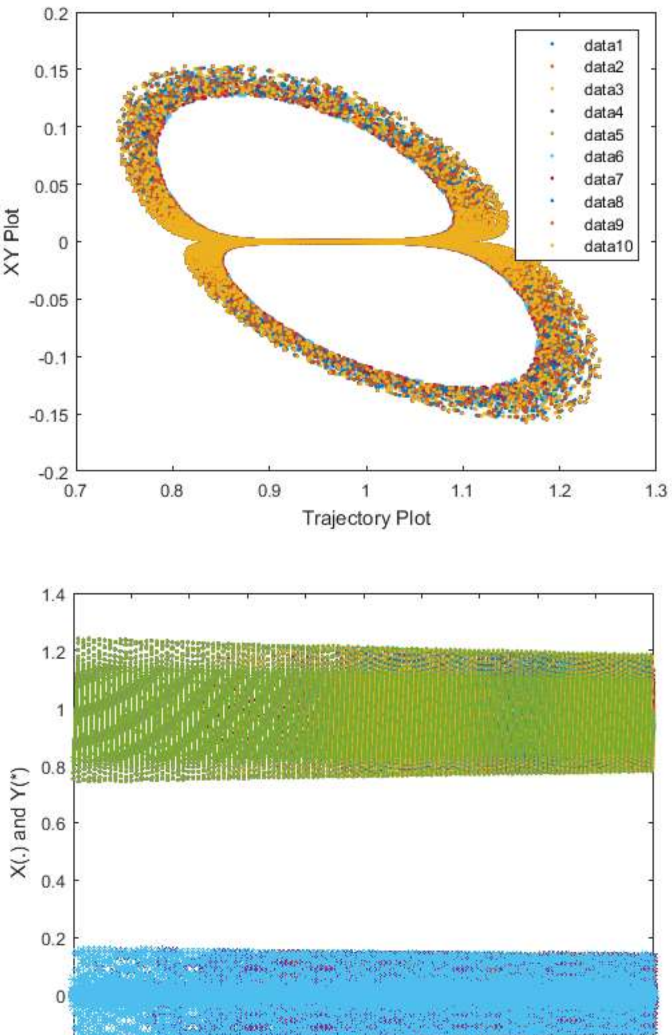
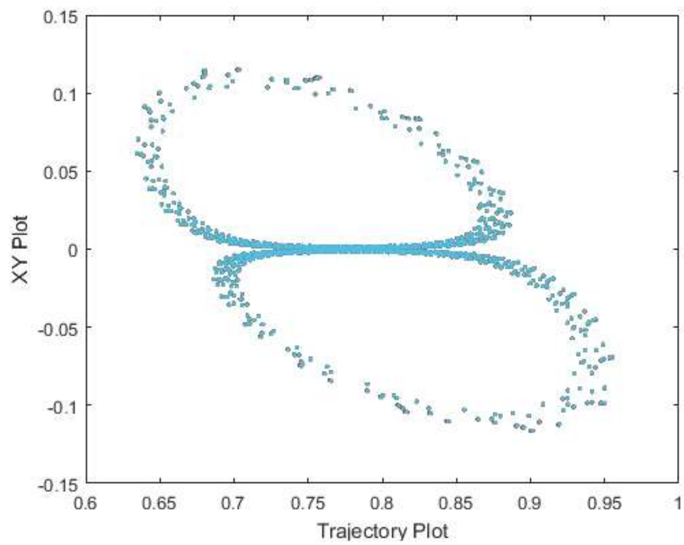


Fig 5.23.2. Limit Cycle (above) and trajectories (below) for the parameters $k = 466, m = -362$ & $c = AM(k, m)$.



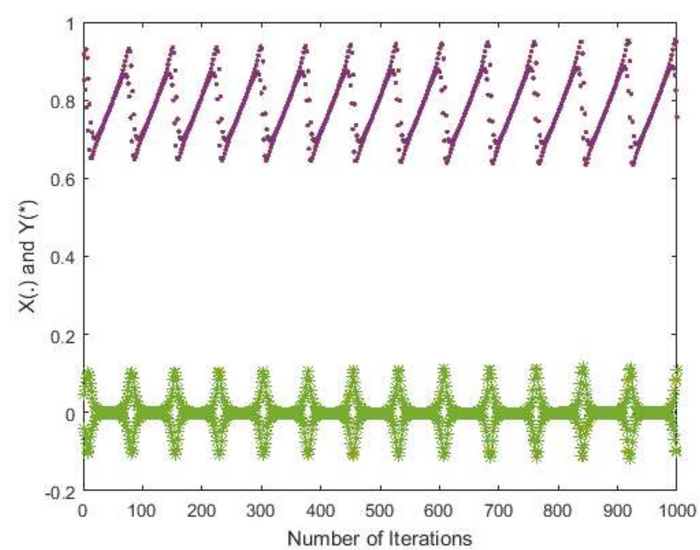


Fig 5.23.3. Limit Cycle (above) and trajectories (below) for the parameters $k = 588, m = -380$ & $c = AM(k, m)$.

Here one more example of parameters $k = 32, m = -23$ and $c = AM(k, m)$ was provided such that the Eqs (5.5-5.6) possesses periodic two solutions. The periodic solution is $(19.066, 0.5589)$ and $(19.8906, -0.2738)$ which is depicted in Fig 5.24.

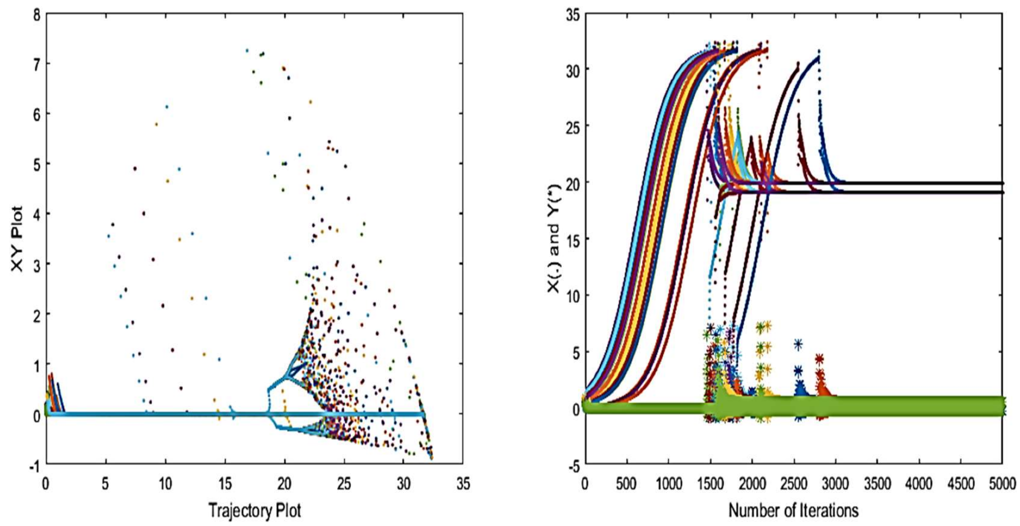


Fig 5.24. Plot of Period 2 trajectories for ten different initial values.

Here another example of parameters $k = 386, m = 410$ and $c = AM(k, m)$ such that the Eqs (5.5-5.6) possess period two solution is given. The periodic solution is $(0.0147, 3.3743)$ and $(-0.0182, -3.3778)$ which is depicted in Fig 5.25.

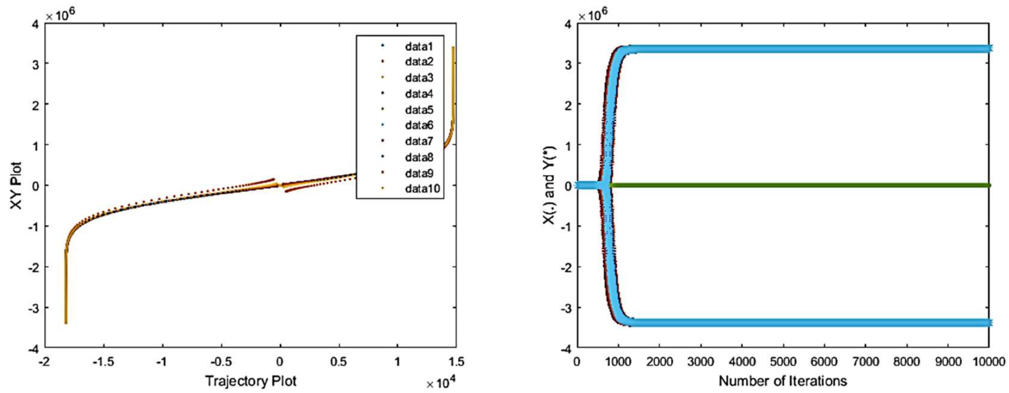


Fig 5.25. Plot of Period 2 trajectories for ten different initial values.

Here one example of parameters $k = 28, m = -288$ and $c = AM(k, m)$ such that the Eqs (5.5-5.6) possesses period two solution was given. The periodic solution is $(1.7762, 0.0499)$ and $(1.9061, -0.0481)$ which is depicted in Fig 5.26.

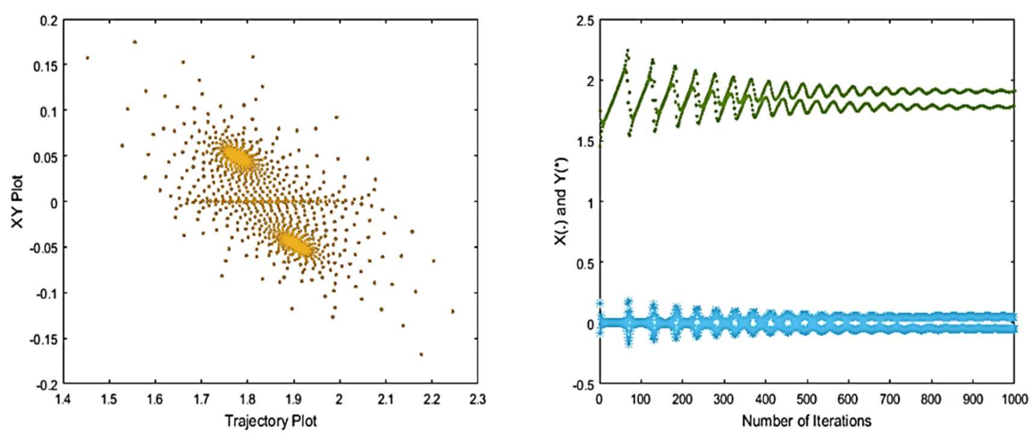


Fig 5.26. Plot of period 2 trajectories for ten different initial values.

The summary of dynamics of Rosenzweig-Macarthur models by means of comparison between the equations (5.3-5.4) and (5.5-5.6) is shown in the following section.

5.5 Comparison of Dynamics

This section deals with the quantitative comparison of the dynamics of the Rosenzweig-Macarthur model as stared in Eqs (5.3-5.4) and Eqs (5.5-5.6). In transiting from the Rosenzweig-Macarthur model Eqs (5.3-5.4) with real parameters to the model Eqs (5.5-5.6), foremost change is in the count of fixed points (from three to four). The points $(0,0)$ and $(k, 0)$ remain fixed for both the models. The following are some qualitative comparative observations:

- The fixed point $(0,0)$ is not attracting for both the models
- In both the systems, the repelling cases for the fixed points are sensitive to initial value especially to value of y.
- The fixed point $(0,0)$ is repelling under the same conditions as derived.
- The fixed point $(0,0)$ exhibits saddle behavior in both the models.
- The fixed point $(k, 0)$ has a large basin of attraction in the system Eqs (5.3-5.4) while for the system Eqs (5.5-5.6) the basin of attraction of fixed point

$$\left(\frac{1}{2}\left(-\sqrt{k}\sqrt{4c + km^2 - 2km + k} + k(-m) + k\right), \frac{\sqrt{k}m\sqrt{4c + km^2 - 2km + k} - 2c - km^2 + km}{2c}\right)$$

  is large.

- In the system (5.3-5.4) there does not exists any attracting limit cycle for any negative parameters (c, k, m) but contrary to it the system equations (5.5-5.6) consists of attracting limit cycles for negative parameters also.

Here the focus is on specific examples for quantitative comparison of dynamics of the model equations Eqs. (5.3-5.4) and Eqs. (5.5-5.6).

Taking parameters $c = 1, k = 5$ and $m = 2$ for both the models studied the model to see the trajectories and corresponding two dimensional plots in Fig. 5.27 for initial values $x_0 = 1, y_0 = 2$.

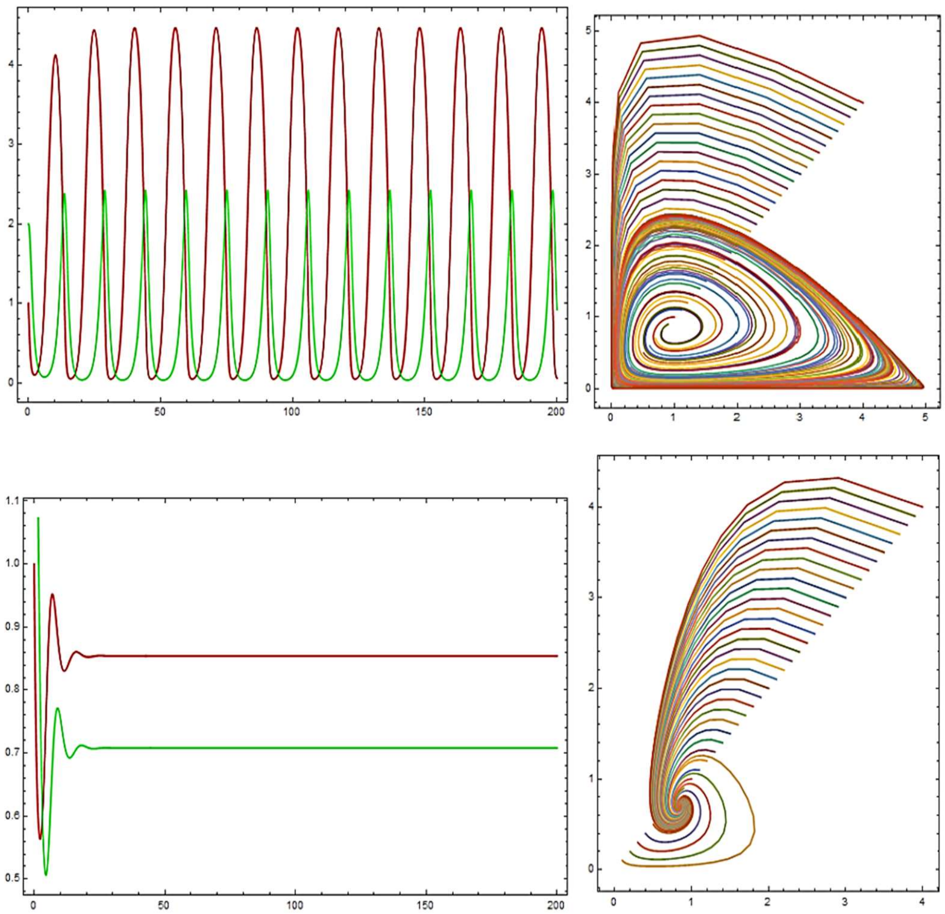


Fig 5.27. Plot of trajectories and two dimensional plot for parameters $c = 1, k = 5$, and $m = 2$ for Eqs (5.3-5.4) (below) and Eqs (5.5-5.6) (above).

It is seen in that for the equations (5.3-5.4) the trajectory is convergent and for the Eqs (5.5-5.6) the trajectory is periodic as depicted in Fig 5.27.

The parametric two dimensional plots were drawn where the parameters x and y are varying from $0\ to\ 2.5$ and $0\ to\ 1.5$ respectively with 200 iterations for both the models as shown in Fig 5.28.

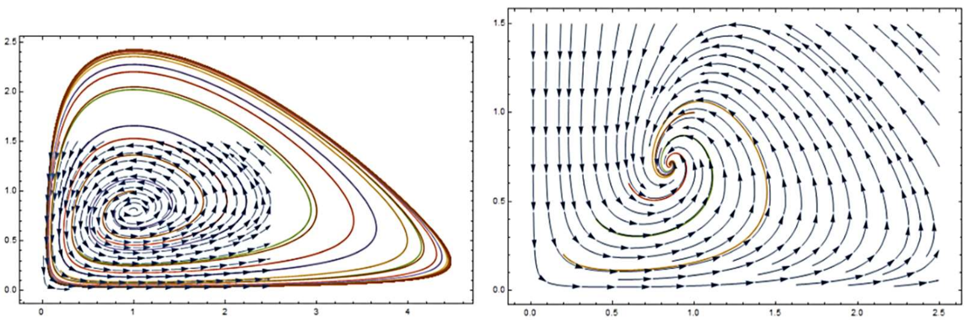


Fig 5. 28. Parametric two dimensional plots for the parameters $c = 1, k = 5$ and $m = 2$ for the Eqs (5.3-5.4) and Eqs (5.5-5.6).

As expected, for the ranges of different initial values (x, y) for the parameters $c = 1, k = 5$, and $m = 2$, trajectories are making a limit cycle in the model Eqs (5.3-5.4) and convergent for the model Eqs (5.5-5.6) as observed left and right images of Fig. 5.28 respectively.

Now the same was done with two different values of the parameter $k = 2$, and $k = 7$ with $c = 1$, $m = 2$ for both the model Eqs (5.3-5.4) and Eqs (5.5-5.6) which is shown in Fig 5.29.

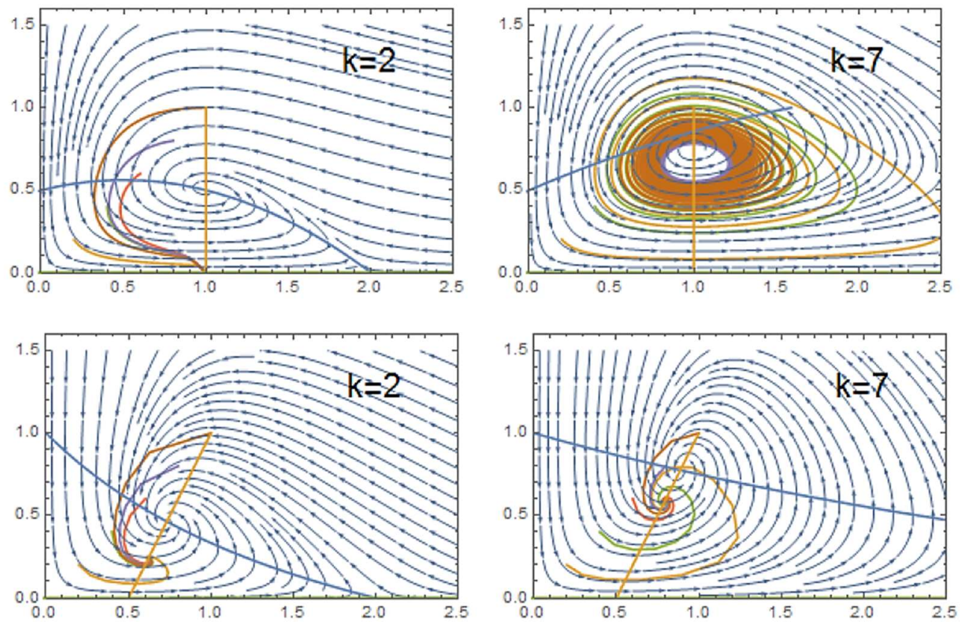


Fig 5.29. Parametric two-dimensional plot for the parameters $c = 1, k = 2$ & 7 and $m = 2$ for Eqs (5.3-5.4) (above) and Eqs (5.5-5.6) (below).

5.6 Conclusion

The dynamics of Rosenzweig-McArthur model under both prey and predator perspectives are studied. Local stability analysis was conducted at fixed points. The limit cycle solutions are discussed. Through examples, quantitative comparison of dynamics of the models is made.

CHAPTER 6. CONCLUSION

In the present work, the first objective was to study a three-dimensional chaotic cancer model. A model was proposed to study the dynamic nonlinear interactions between tumor, healthy host, and effector immune cells where all the parameters are considered as non-negative real numbers. The study explored the dynamics of the model computationally. The study revealed that the parameters of the system are very sensitive and for a range of positive values of the parameters, the system exhibited chaos. Some of the similar previous studies hinted about the existence of chaotic behavior but in this work, this fact was established by using Hurst exponent and fractal dimension.

In the study of Nicholson Bailey models, the dynamics of the model equations are reinvestigated by considering the parameters of the system as real numbers instead of restricting them only to positive real numbers as in the case of other significant studies of the Nicholson Bailey models. Our study revealed that the system exhibits all sorts of dynamics such as chaotic, periodic, locally stable, and unstable equilibriums.

In addition to this, a scaling model was considered by including scaling factors in the Nicholson Bailey model and studied its dynamics. For the scaled system, it was obtained that the fixed points range over two-dimensional plane, unlike the original unscaled model. It was observed that the behavior of the system changed drastically with changes in the system parameters. Several bifurcations were observed and plotted. The impact of scaling factors on the sensitivity of chaotic behavior was established.

Uniformly distributed noise is added to the Nicholson Bailey model and the stability of the system is studied. This extensive study of the Nicholson Bailey models produced several significant results in terms of periodicity and chaotic behavior. The bilateral symmetry in the dynamics of the system was noticed by extending the range of parameters from positive real numbers to entire real line.

The discrete versions of the Rosenzweig-McArthur model under both prey and predator perspectives were studied. Local stability analysis was conducted at fixed points. It was observed that the system possess attracting limit cycles under predator perspective for negative parameters and no such attracting limit cycles for negative parameters were observed under prey perspective.

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   Research Article:
   https://doi.org/10.14445/22315373/IJMTT-V53P544

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   Journal: International Journal of Mathematics Trends and Technology (IJMTT),
   Vol. 53, no.5, pp. 353-368, 2018.

2. **Publication 1: SCI publication.**
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